## Mathematical Appendix

## A. Second Moments

Many of the asset pricing relationships include the second statistical moment between two random variables. Thus, we will look here at different ways of expressing the second moment: variance and covariance, correlation coefficient, and beta. We will also derive a linearity rule of manipulating covariances and consider some other convenient properties of covariances.

## Covariance and Related Definitions

Consider two random variables, $X$ and $Y$, with means (or "expected values" or "first moments") of $\mu_{X}$ and $\mu_{Y}$. Then the Covariance between $X$ and $Y$ is given by:

$$
\begin{equation*}
\operatorname{Cov}(X, Y) \equiv \sigma_{X Y} \equiv E\left[\left(X-\mu_{X}\right)\left(Y-\mu_{Y}\right)\right] . \tag{A1}
\end{equation*}
$$

The Variance of a random variable $X, \operatorname{Var}(X)$, is a special case of the covariance, in which the random variables, here $X$ and $Y$, are identical. The Standard Deviation is simply the square root of the variance. Thus:

$$
\begin{equation*}
\sigma_{x} \equiv[\operatorname{Var}(X)]^{1 / 2} \equiv\left[E\left(X-\mu_{X}\right)^{2}\right]^{1 / 2} \tag{A2}
\end{equation*}
$$

To normalize the covariance such that its value must lie between -1 and +1 , we define the Correlation Coefficient between $X$ and $Y$ as:
(A3) $\quad \rho_{X Y}=\frac{\sigma_{X Y}}{\sigma_{X} \sigma_{Y}}$.
Another concept related to the covariance is that of the slope of a simple OLS regression line $Y="+\$ X$ :
(A4) $\quad \beta \equiv \beta_{X Y}=\frac{\sigma_{X Y}}{\sigma_{X}^{2}}$.
Clearly, it follows from (A3) and (A4) that $\rho_{X Y}=\beta_{X Y}\left(\sigma_{X} / \sigma_{Y}\right)$.

## Manipulating Covariances

Expanding the definition in (A1) we can derive that
$\sigma_{X Y}=E(X Y)-\mu_{Y} E(X)-\mu_{X} E(Y)+\mu_{X} \mu_{Y}$. Thus given that $\mu_{Z} \equiv E(Z)$ for any random variable we then find that:

$$
\begin{equation*}
\sigma_{X Y}=E(X Y)-\mu_{X} \mu_{Y} \quad \text { or } \quad E(X Y)=\sigma_{X Y}+\mu_{X} \mu_{Y} \tag{A5}
\end{equation*}
$$

Next we prove the linearity of the covariance operator. Define $X=a X_{1}+b X_{2}$. Then we can write $\operatorname{Cov}\left(a X_{1}+b X_{2}, Y\right)=E\left\{\left[a\left(X_{1}-\mu_{X_{1}}\right)+b\left(X_{2}-\mu_{X_{2}}\right)\right]\left(y-\mu_{Y}\right)\right\}$ as follows easily from (A1). Thus,
(A6)

$$
\operatorname{Cov}\left(a X_{1}+b X_{2}, Y\right)=a \operatorname{Cov}\left(X_{1}, Y\right)+b \operatorname{Cov}\left(X_{2}, Y\right)
$$

Or, equivalently, $\sigma_{X Y}=a \sigma_{X_{1} Y}+b \sigma_{X_{2} Y}$. So, the covariance is a linear operator. Note that as an application of (A6) we can write $\operatorname{Var}(a X)=\operatorname{Cov}(a X, a X)=a^{2} \operatorname{Var} X$. Also note that $\operatorname{Cov}(X+b, Y)=\operatorname{Cov}(X, Y)$.

Lastly, $\operatorname{Var}(X+Y) \equiv \operatorname{Cov}(X+Y, X+Y)=\operatorname{Cov}(X, X+Y)+\operatorname{Cov}(Y, X+Y)$. Thus,

$$
\begin{equation*}
\operatorname{Var}(X+Y) \equiv \operatorname{Var}(X)+2 \operatorname{Cov}(X, Y)+\operatorname{Var}(Y) \tag{A7}
\end{equation*}
$$

## Covariance and Matrix Notation

If we assume $n$ equally likely possible outcomes $\left(x_{i}, y_{i}\right)$ then (A1) becomes:

$$
\begin{equation*}
\operatorname{Cov}(X, Y)=\frac{1}{n}\left\{\left[\sum_{i=1}^{n} x_{i}-\left(\sum_{i=1}^{n} x_{i} / n\right)\right]\left[y_{i}-\left(\sum_{i=1}^{n} y_{i} / n\right)\right]\right\} \tag{A8}
\end{equation*}
$$

Consider $n$ random variables $X_{i}$. Matrix $\Sigma$ is defined as the variance-covariance matrix of the $X_{i}$, having as its $(i, j)$ element $\operatorname{Cov}\left(X_{i}, X_{j}\right)$ and accordingly as its $(i, i)$ element $\operatorname{Var}\left(X_{i}\right)$. If we now define the vector of random variables $X_{i}$ as $\mathbf{x}$ and define $X$ as the weighted sum of the random variables $X_{i}$, with weights $s_{i}$ in column vector notation $\mathbf{s}$, then
$X=\mathbf{s}^{\mathbf{T}} \mathbf{x}$ and
(A9) $\quad\left[\operatorname{Cov}\left(X, X_{1}\right), \operatorname{Cov}\left(X, X_{2}\right), \ldots \operatorname{Cov}\left(X, X_{n}\right)\right]=\mathbf{s}^{\mathbf{T}} \Sigma$.

Additionally,
(A10) $\quad \operatorname{Var}(X)=\mathbf{s}^{\mathbf{T}} \boldsymbol{\Sigma} \mathbf{s}$.

Similarly, an expression for the covariance of two random variables $X_{A}$ and $X_{B}$ which are linear combinations of the $X_{i}$ with weight vectors $\mathbf{s}_{\mathrm{A}}$ and $\mathbf{s}_{\mathbf{B}}$ can be obtained as:

$$
\begin{equation*}
\operatorname{Cov}\left(X_{A}, X_{B}\right)=\mathbf{s}_{\mathbf{A}}^{\mathbf{T}} \boldsymbol{\Sigma} \mathbf{s}_{\mathbf{B}} \tag{A11}
\end{equation*}
$$

To connect the expression of $\beta_{X Y}$ in (A4) to the matrix notation used in regression analysis, define $\mathbf{X}$ as the matrix having as its first column a constant and in the following columns the random variables $X_{i}$ and as its rows each of the possible outcomes for the constant (1s of course) and the $X_{i}$. Then the vector of OLS regression constant and
slope coefficients is found as:
(A12) $\boldsymbol{\beta}=\left(\mathbf{X}^{\prime} \mathbf{X}\right)^{-\mathbf{1}} \mathbf{X}^{\prime} \mathbf{y}$,
where $\mathbf{y}$ represents the possible outcomes for the random variable $Y$. It is somewhat tedious but useful to check that, if we consider a simple regression, including just a constant and one independent variable $X$, that the slope coefficient $\beta_{X Y}$, the second coefficient in the $\beta$ vector, is given as in (A4).

## B. Some Useful Regression Analogies

Least Squares with Two Independent Variables

Consider three random variables $Y, X_{l}$, and $X_{2}$. Suppose we want to related these three variables linearly such that:
(B1) $\quad Y=\alpha+\beta_{1} X_{1}+\beta_{2} X_{2}+\varepsilon, \quad E(\varepsilon)=E\left(\varepsilon X_{1}\right)=E\left(\varepsilon X_{2}\right)=0$.

We show by construction that this is always possible by choosing the coefficients $\alpha, \beta_{1}, \beta_{2}$ such that

$$
\begin{equation*}
E\left(\varepsilon^{2}\right)=E\left[\left(Y-\alpha-\beta_{1} X_{1}-\beta_{2} X_{2}\right)^{2}\right], \tag{B2}
\end{equation*}
$$

is minimized. The first-order conditions for this minimization with respect to the three coefficients are:

$$
\begin{equation*}
E(Y)-\alpha-\beta_{1} E\left(X_{1}\right)-\beta_{2} E\left(X_{2}\right)=0 \tag{B3}
\end{equation*}
$$

$$
\begin{align*}
& E\left[X_{1}\left(Y-\alpha-\beta_{1} X_{1}-\beta_{2} X_{2}\right)\right]=0  \tag{B4}\\
& E\left[X_{2}\left(Y-\alpha-\beta_{1} X_{1}-\beta_{2} X_{2}\right)\right]=0 \tag{B5}
\end{align*}
$$

Using the notation from Appendix A, we can write (B3) as:

$$
\begin{equation*}
\alpha=\mu_{Y}-\beta_{1} \mu_{X_{1}}-\beta_{2} \mu_{X_{2}} \tag{B6}
\end{equation*}
$$

and further, using (A5) and (B6), we can write (B4) and (B5) as:
(B7) $\beta_{1}=\frac{\sigma_{X_{1} Y}-\beta_{2} \sigma_{X_{1} X_{2}}}{\sigma_{X_{1}}^{2}}, \quad \beta_{2}=\frac{\sigma_{X_{2} Y}-\beta_{1} \sigma_{X_{1} X_{2}}}{\sigma_{X_{2}}^{2}}$.

Solving for $\beta_{1}$ and $\beta_{2}$ in (B7) yields:
(B8) $\quad \beta_{1}=\frac{\sigma_{Y X_{1}} \sigma_{X_{2}}^{2}-\sigma_{Y X_{2}} \sigma_{X_{1} X_{2}}}{\sigma_{X_{1}}^{2} \sigma_{X_{2}}^{2}-\sigma_{X_{1} X_{2}}^{2}}, \quad \beta_{2}=\frac{\sigma_{Y X_{2}} \sigma_{X_{1}}^{2}-\sigma_{Y X_{1}} \sigma_{X_{1} X_{2}}}{\sigma_{X_{1}}^{2} \sigma_{X_{2}}^{2}-\sigma_{X_{1} X_{2}}^{2}}$.

Thus, for any three random variables $Y, X_{1}$, and $X_{2}$ (given that the first and second moments exist), we can relate them linearly as in (B2), with coefficients given in (B6) and (B8), and such that $\varepsilon$ has zero mean and is uncorrelated with $X_{1}$, and $X_{2}$. If, furthermore, the three variables are normally distributed then $\varepsilon$ is normally distributed as well and is independent of $X_{1}$, and $X_{2}$.

## Instrumental Variables

Suppose we have a linear relation between random variables $X$ and $Y$ given as:
(B9) $\quad Y=\alpha+\beta X+\varepsilon, \quad E(\varepsilon X) \neq 0$.

We would like to find an estimate for $\beta$ but $X$ and $\varepsilon$ are correlated (either because of measurement error in $X$, omitted variables, or because of a simultaneity problem). OLS will provide biased and inconsistent estimates. We should use an instrument $Z$ for $X$ that is uncorrelated with $\varepsilon$ and preferably highly correlated with $X$.

To estimate $\beta$ with the use of the instrument $Z$ we proceed in two steps. First find the decomposition:

$$
\text { (B10) } \quad X=\gamma+\delta Z+\eta, \quad E(\eta)=E(\eta Z)=0, \quad \delta=\sigma_{X Z} / \sigma_{Z}^{2}
$$

The part of $X$ that is correlated with $\varepsilon$ is now incorporated in $\eta$ only since $Z$ is by assumption uncorrelated with $\varepsilon$. Thus we only use the part $\gamma+\delta Z$ in the regression. Substituting equation (B10) into equation (B9) yields:
(B11) $\quad Y=\alpha+\beta \gamma+(\beta \delta) Z+\beta \eta+\varepsilon, E(\eta)=E(\varepsilon)=E(\eta Z)=E(\varepsilon Z)$.

We can obtain this decomposition as before since $Z$ is uncorrelated with $\varepsilon$ by assumption and with $\eta$ by construction. The slope coefficient then becomes:
(B12) $\quad \beta \delta=\frac{\sigma_{Y Z}}{\sigma_{Z}^{2}} \rightarrow \quad \beta=\frac{\sigma_{Y Z}}{\delta \sigma_{Z}^{2}}=\frac{\sigma_{Y Z}}{\sigma_{X Z}}$.

The estimate of $\beta$ obtained in this way is still biased (due to the presence of $\beta$ in the error term) but is now consistent.

## C. Stein's Lemma and Generalization

## Statement and Proof of Stein's Lemma

"Stein's Lemma" provides a linearization result for covariances under a normality assumption when one argument is a (possible) non-linear function of a normal variable. As such it is potentially of great value in many finance applications. Nevertheless it is not applied frequently. Huang and Litzenberger (1988) employ it without proof, referring to a paper by Rubinstein (1974) and crediting the statistician Stein for the result. Neither reference is helpful in providing further information about the Lemma. The Rubinstein application is also without proof but appears to be the first application of the result in the economics literature. The reference to Stein is not specific and I was unable to find a statement and proof of the Lemma. Ingersoll (1987) however proves the result without calling it Stein's Lemma. His proof is given below.

Stein's Lemma states that:
(C1) $\operatorname{Cov}[X, h(Y)]=E\left[h_{Y}(Y)\right] \operatorname{Cov}(X, Y)$,
where $X$ and $Y$ are bivariate normal; $h()$ is a differentiable function; and the subscript $Y$ indicates the (partial) derivative.

The proof is as follows. First use a standard decomposition:
(C2) $\quad X=\alpha+\beta_{Y X} Y+\varepsilon ; \quad E(\varepsilon)=E(\varepsilon Y)=0, \quad \beta_{Y X}=\sigma_{X Y} / \sigma_{Y}^{2}$.

Such a decomposition exists for any two random variables and is in fact accomplished by an OLS regression. For normal distributions, the zero correlation between $\varepsilon$ and $Y$ implies independence. Hence, $\operatorname{Cov}[\varepsilon, h(Y)]=0$. Thus, as follows from the linearity property (A6) and (C2):
(C3) $\quad \operatorname{Cov}[X, h(Y)]=\beta_{Y X} \operatorname{Cov}[Y, h(Y)]$.

Given equation (A1) for a continuous distribution and since $E\left\{\left(Y-\mu_{Y}\right) E[h(Y)]\right\}=0$, we can write the covariance between $Y$ and $h(Y)$ as:
(C4) $\operatorname{Cov}[Y, h(Y)]=\int_{-\infty}^{\infty}\left(Y-\mu_{Y}\right) h(Y) f(Y) d Y$,
where $\mu_{Y}$ is the mean of $Y ; f(Y)$ is its (normal) density function, which is given as

$$
\text { (C5) } \quad f(Y)=\frac{1}{\sigma_{Y}(2 \pi)^{1 / 2}} \exp \left[-\frac{\left(Y-\mu_{Y}\right)^{2}}{2 \sigma_{Y}^{2}}\right] \text {. }
$$

From (C5), differentiation yields:
(C6) $\quad \frac{d f(Y)}{d Y}=-\frac{Y-\mu_{Y}}{\sigma_{Y}^{2}} f(Y)$.
Substituting (C6) into (C4) gives:
(C7) $\operatorname{Cov}[Y, h(Y)]=-\sigma_{Y}^{2} \int_{-\infty}^{\infty} h(Y) d f(Y)$.
now integrate the right-hand side of (C7) by parts:
(C8) $\operatorname{Cov}[Y, h(Y)]=-\sigma_{Y}^{2}\left(-\int_{-\infty}^{\infty} h_{Y}(Y) d Y+\left.[h(Y) f(Y)]\right|_{-\infty} ^{\infty}\right)$.
The last term vanishes at both limits, since the normal density converges to zero at the limits, as long as $h(Y)$ does not go to infinity too fast; or, in other words, as long as $h(Y)=o\left[\exp \left(Y^{2}\right)\right]$. Thus,
(C9) $\operatorname{Cov}[X, h(Y)]=\beta_{Y X} \operatorname{Cov}[Y, h(Y)]=\beta_{Y X} \sigma_{Y}^{2} E\left[h_{Y}(Y)\right]$.

Stein's Lemma in (C1) then follows directly from (C9) plus the definition of $\beta_{Y X}$ in (C2).

## A Multivariate Generalization of Stein's Lemma

The bivariate generalization of Stein's Lemma is stated as:
(C10) $\operatorname{Cov}[X, h(Y, Z)]=E\left[h_{Y}(Y, Z)\right] \operatorname{Cov}(X, Y)+E\left[h_{Z}(Y, Z)\right] \operatorname{Cov}(X, Z)$.

Here $X, Y$, and $Z$ are multivariate normal and $h()$ is differentiable in its two arguments. Given the bivariate generalization, the statement and proof of the multivariate generalization is obvious so we will limit ourselves to discussing the bivariate case only.

We can express $X$ in terms of the other random variables, using the standard decomposition:

$$
\begin{aligned}
& \text { (C11) } \quad X=\alpha+\beta_{Y} Y+\beta_{Z} Z+\varepsilon ; \quad E(\varepsilon)=E(\varepsilon Y)=E(\varepsilon Z)=0, \\
& \quad \beta_{Y}=\frac{\sigma_{Y X} \sigma_{Z}^{2}-\sigma_{Z X} \sigma_{Y Z}}{\sigma_{Y}^{2} \sigma_{Z}^{2}-\sigma_{Y Z}^{2}}, \quad \beta_{Z}=\frac{\sigma_{Z X} \sigma_{Y}^{2}-\sigma_{Y X} \sigma_{Y Z}}{\sigma_{Y}^{2} \sigma_{Z}^{2}-\sigma_{Y Z}^{2}}
\end{aligned}
$$

The slope coefficients are analogous to those in a multivariate regression with two independent variables and a constant as discussed in Appendix B and are given in (B8). From (C11) and the linearity of the covariance operator we can obtain:
(C12) $\operatorname{Cov}[X, h(Y, Z)]=\beta_{Y} \operatorname{Cov}[Y, h(Y, Z)]+\beta_{Z} \operatorname{Cov}[Z, h(Y, Z)]$.

Now consider the covariance terms on the right-hand side of (C12):
(C13) $\operatorname{Cov}[Y, h(Y, Z)]=\operatorname{Cov}\left[Y, h\left(Y, \alpha_{Z}+\beta_{Y Z} Y+\varepsilon_{Z}\right)\right]$,
with similar conditions as in (C2). Next we show a lemma, that:

$$
\begin{equation*}
\operatorname{Cov}\left[Y, g\left(Y, \varepsilon_{Z}\right)\right]=E\left[g_{Y}\left(Y, \varepsilon_{Z}\right)\right] \sigma_{Y}^{2} \tag{C14}
\end{equation*}
$$

Proof: By construction $Y$ and $\varepsilon_{Z}$ are uncorrelated and, given that both are normally distributed, therefore are independent. Thus:
(C15) $\operatorname{Cov}\left[Y, g\left(Y, \varepsilon_{Z}\right)\right]=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left(Y-\mu_{Y}\right) g\left(Y, \varepsilon_{Z}\right) f^{Y}(Y) f^{\varepsilon}\left(\varepsilon_{Z}\right) d \varepsilon d Y$,
where $f^{Y}()$ and $f^{g}()$ are normal density functions. As in (C6):
(C16) $\quad \frac{d f^{Y}(Y)}{d Y}=-\frac{Y-\mu_{Y}}{\sigma_{Y}^{2}} f^{Y}(Y)$.
Substitute (C16) into (C15) gives:

$$
\text { (C17) } \operatorname{Cov}\left[Y, g\left(Y, \varepsilon_{Z}\right)\right]=-\sigma_{Y}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(Y, \varepsilon_{Z}\right) d f^{Y}(Y) f^{\varepsilon}\left(\varepsilon_{Z}\right) d \varepsilon
$$

Integrating by parts the inner integral gives:

$$
\begin{array}{ll}
\text { (C18) } & \operatorname{Cov}\left[Y, g\left(Y, \varepsilon_{Z}\right)\right]= \\
& -\sigma_{Y}^{2} \int_{-\infty}^{\infty}\left(-\int_{-\infty}^{\infty} g_{Y}\left(Y, \varepsilon_{Z}\right) f^{Y}(Y) d Y+\left.\left[g\left(Y, \varepsilon_{Z}\right) f^{Y}(Y)\right]\right|_{Y=-\infty} ^{\infty}\right) f^{\varepsilon}\left(\varepsilon_{Z}\right) d \varepsilon
\end{array}
$$

The final term in parentheses vanishes if $g\left(Y, \varepsilon_{Z}\right)=o\left[\exp \left(Y^{2}\right)\right]$. Equation (C18) can then be written as:
(C19)

$$
\operatorname{Cov}\left[Y, g\left(Y, \varepsilon_{Z}\right)\right]=\sigma_{Y}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{Y}\left(Y, \varepsilon_{Z}\right) f^{Y}(Y) f^{\varepsilon}(\varepsilon) d Y d \varepsilon
$$

which is (C14). Now (C13) becomes:

$$
\text { (C20a) } \operatorname{Cov}[Y, h(Y, Z)]=\left\{E\left[h_{Y}(Y, Z)\right]+\beta_{Y Z} E\left[h_{Z}(Y, Z)\right]\right\} \sigma_{Y}^{2},
$$

and similarly
(C20b) $\operatorname{Cov}[Z, h(Y, Z)]=\left\{E\left[h_{Z}(Y, Z)\right]+\beta_{Z Y} E\left[h_{Z}(Y, Z)\right]\right\} \sigma_{Z}^{2}$.

Returning to (C12), using (C20a) and (C20b), and using the fact that $\beta_{i j}=\sigma_{i j} / \sigma_{i}^{2}$, we obtain:

$$
\begin{gather*}
\operatorname{Cov}[X, h(Y, Z)]=\beta_{Y}\left(E\left[h_{Y}(Y, Z)\right] \sigma_{Y}^{2}+E\left[h_{Z}(Y, Z)\right] \sigma_{Y Z}\right)  \tag{C21}\\
+\beta_{Z}\left(E\left[h_{Z}(Y, Z)\right] \sigma_{Z}^{2}+E\left[h_{Y}(Y, Z)\right] \sigma_{Y Z}\right)
\end{gather*}
$$

Collecting terms yields:
(C22)

$$
\begin{gathered}
\operatorname{Cov}[X, h(Y, Z)]=\left(\beta_{Y} \sigma_{Y}^{2}+\beta_{Z} \sigma_{Y Z}\right) E\left[h_{Y}(Y, Z)\right] \\
+\left(\beta_{Z} \sigma_{Z}^{2}+\beta_{Y} \sigma_{Y Z}\right) E\left[h_{Z}(Y, Z)\right]
\end{gathered}
$$

The definitions of $\beta_{Y}$ and $\beta_{Z}$ in (C11) then produce the desired result as stated in (C10).

## D. STOCHASTIC DYNAMIC PROGRAMMING

## Problem Statement

Consider the following stochastic dynamic decision problem with a finite horizon:
(D1) $\quad V\left(x_{s}, s\right)=\begin{aligned} & \max \\ & \left\{u_{t}\right\}_{t=s}^{T-1}\end{aligned} E_{s}\left[\sum_{t=s}^{T-1} \beta^{t-s} f\left(x_{t}, u_{t}\right)\right]$,
(D2) Subject to: $x_{t+1}=g\left(x_{t}, u_{t}, \varepsilon_{t+1}\right)$ and $f\left(x_{T}, \cdot\right)=B\left(x_{T}\right)$

Here $f()$ represents the objective function for each period. The agent is assumed to maximize the expected discounted present value of the objective for each period, where the discount factor $\beta$ lies between zero and one. The choice variable in each period is $u_{t}$ is called the control variable whereas the current status of the system is described by the state variable $x_{t}$. The constraint describes how the state variable changes over time and is called the equation of motion. The term g indicates a shock that stochastically impacts the motion of the state variable.

The objective ("bequest" $B()$ ) in the final period $T$ depends solely on the value of the state variable at that point. $V()$ would represent the "indirect utility function" if the objective were utility. In general it is called the value function. Since the control variable is "maximized out" it depends only on the variables that summarize the current situation: time and the state variable. Lastly, the problem described here is not the most general. More general problems may easily be treated analogously. For instance, it is straightforward to think of $x_{t}$ and $u_{t}$ as vectors of state

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and control variables.
A very rigorous discussion of problems of this type can be found in Stokey and Lucas (1989). Additionally, Chapter 1 and the Appendix in Sargent (1987) provide additional material that help in applying the stochastic dynamic programming technique to economic problems. Sargent's treatment is not as rigorous as that of Stokey and Lucas, but quite a bit easier. The discussion below is loosely based on Sargent (1987).

## Backward Induction and the Bellman Equation

The way to analyze the above problem is to "start at the end": Since the current decision affects the state for the next period, it is important to know how important this state is for future outcomes. But that is only clear if it is known what decisions will be made in the future. So the decision rules for later periods must be known in earlier periods in order to take optimal decisions. Accordingly, the final decision should be examined first.

In the above decision problem we can write at the time of the final decision:
(D3) $\quad V\left(x_{T-1}, T-1\right)=\max _{u_{T-1}}\left[f\left(x_{T-1}, u_{T-1}\right)+\beta E_{T-1} B\left(x_{T}\right)\right]$,
(D4) Subject to: $x_{T}=g\left(x_{T-1}, u_{T-1}, \varepsilon_{T}\right)$.

Once $V\left(x_{T-1}, T-1\right)$ has been obtained, in effect the decision rule for the final period has been incorporated in the value function. It is then straightforward to go back one more period and obtain:

$$
\begin{equation*}
V\left(x_{T-2}, T-2\right)=\max _{u_{T-2}}\left[f\left(x_{T-2}, u_{T-2}\right)+\beta E_{T-2} V\left(x_{T-1}, T-1\right)\right] \tag{D5}
\end{equation*}
$$

$$
\begin{equation*}
\text { Subject to: } \quad x_{T-1}=g\left(x_{T-2}, u_{T-2}, \varepsilon_{T-1}\right) \text {. } \tag{D6}
\end{equation*}
$$

By induction we can then find for any time period $t$ :
(D7) $\quad V\left(x_{t}, t\right)=\begin{array}{r}\max \\ u_{t}\end{array} \quad\left[f\left(x_{t}, u_{t}\right)+\beta E_{t} V\left(x_{t+1}, t+1\right)\right]$,
(D8) Subject to: $x_{t+1}=g\left(x_{t}, u_{t}, \varepsilon_{t+1}\right)$, and $V\left(x_{T}, T\right)=B\left(x_{T}\right)$.

The decision problem in equations (D1) and (D2) is equivalent to the decision problem in equations (D7) and (D8). The recursive method of solving a finite horizon problem in this way is called Backward Induction. Equation (D7) is named the Bellman equation after the "inventor" of dynamic programming. It relates the current maximum value to the maximum for the objective based on how the current decision affects the current objective and next period's state; the effect on next period's state is evaluated for how if affects the maximum value for the next period. Effectively, once the value function is known, the future is collapsed into one period.

Second-order conditions for each decision are just that the right-hand side of equation (D7) is concave. One

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set of sufficient conditions is that the objective $f()$ is strictly concave in both the state variable and the control variable, and that the constraint set described by the equation of motion is convex. Then the value function will be strictly concave in the state variable and second-order conditions will hold for the decision in each period.

## The Infinite Horizon Problem

If we let $T \rightarrow \infty$ in decision problem (D1) and (D2) then the bequest function becomes irrelevant (because $\beta^{\mathrm{T}} B()$ goes to zero). The problem now changes from period to period only in as far as the state variable changes. Note that we need to divide the value function by $\beta^{t}$ in order to render it independent of time. Thus we have:
(D9) $V\left(x_{t}\right)=\max _{u_{t}}\left[f\left(x_{t}, u_{t}\right)+\beta E_{t} V\left(x_{t+1}\right)\right]$,
(D10) Subject to: $x_{t+1}=g\left(x_{t}, u_{t}, \varepsilon_{t+1}\right)$.

This problem is called autonomous as it does not depend on time explicitly. Second-order conditions will hold under the same conditions as in the finite-horizon case, which also imply that the value function is concave. If, in addition, $f()$ is differentiable then the value function is differentiable as well.

Clearly, it is now impossible to apply the method of backward induction. To solve the decision problem, first obtain the first-order condition after substituting equation (D10) into (D9):

$$
\begin{equation*}
f_{u}\left(x_{t}, u_{t}\right)=-\beta E_{t}\left[g_{u}\left(x_{t}, u_{t}, \varepsilon_{t+1}\right) V_{x}\left(x_{t+1}\right)\right] \tag{D11}
\end{equation*}
$$

As there is no terminal condition such as the bequest function in this infinite horizon model, we need to impose a transversality condition instead. This is essentially the first-order condition for the "final" period. Consider the following very informal derivation. In a final period the future should become unimportant so that the expected discounted value $V_{x}()$ should converge to zero. Alternatively, consider that at the final period the value of the state variable should be zero. Thus the state variable evaluated at its marginal impact on the value function should be zero. Then equation (D11) would be something like:
(D12) $\quad \lim _{T \rightarrow \infty} \beta^{T} E_{t}\left[x_{T} V_{x}\left(x_{T}\right)\right]=0$.
Note that normally we can divide both sides of the first-order condition by the discount factor as is done in (D11). However, in the case of the transversality condition, the discount factor helps to make the term converge to zero, as long as the marginal value of the state variable does not go to infinity. The term is multiplied by the state variable since the impact of the state variable can be zero if either its quantity is zero or its (marginal) value is zero. The transversality condition can take several different forms; the current version is sufficient to imply an optimum together with the firstorder condition. In practice the transversality condition will hold automatically in most typical decision problems. So it is not needed in constructing a solution but can just be verified to hold after a solution is obtained.

An additional equation that is often useful in describing the implications of the decision model is the envelope
condition obtained by taking the derivative of the value function with respect to the state variable. As implied by the envelope theorem, in taking the derivative the effect of the state variable on the value of the optimal control variable can be ignored. Thus:
(D13) $V_{x}\left(x_{t}\right)=f_{x}\left(x_{t}, u_{t}^{*}\right)+\beta E_{t}\left[g_{x}\left(x_{t}, u_{t}^{*}, \varepsilon_{t+1}\right) V_{x}\left(x_{t+1}\right)\right]$,
where a "*" indicates an optimal value for the control variable. In many cases, the envelope condition can be used together with the first-order condition to eliminate the expectations term and obtain some direct insights from the resulting expression.

## Solving the Infinite Horizon Problem

In many cases, no closed-form solution for the decision rule and for the value function can be obtained. If a closed-form solution exists, however, two different methods can be used to find this solution. First, the "guess and verify" method of undetermined coefficients: Guess a functional form for the value function and/or the decision rule. Then verify that (a) the first-order condition, and (b) the envelope condition or the Bellman equation, hold for your guess. If you can find parameters for your conjectured functional form such that (a) and (b) hold as identities you need only check the transversality condition and you are done. The solution and value function are unique and you have just identified each. Note that using the envelope condition rather than the Bellman equation for condition (b) is usually easier but that a (usually unimportant) constant in the value function remains unidentified. Solutions are usually difficult to obtain unless the problem is linear-quadratic, in which case the value function becomes quadratic, or unless the problem has a power function objective and linear or power function constraint, in which case the value function tends to inherit the power functional form.

The second solution method is to start with any imagined bequest function (such as zero) and start solving the problem by backward induction. There is a theorem (informally proven in our previous discussion) stating that, in the limit, the value function obtained in this way must be equal to the value function for the infinite horizon problem, irrespective of the bequest function that you used. In practice, after a few iterations you may discern a pattern that allows you to guess the value function and employ the first method.

## A Linear Equation of Motion

Often the decision problem simplifies significantly if the equation of motion is assumed to be linear. In many applications the equation of motion is naturally linear in the control and the state variable. For instance when wealth is the state variable its next period value will be the initial wealth minus the control (consumption) times a market return. For equations of motion that are linear in state and control variable and multiplicative in the same random variable, it is always possible to combine equations (D13) and (D11) to eliminate the expectations term.

For equations of motion, and in specific a linear equation of motion it is often necessary to impose an additional condition that is called the isoperimetric condition or, as Sargent (1987) calls it, the no-Ponzi-games condition. For the example of wealth, for instance, we have:
(D14) $\quad W_{t+1}=R_{t+1}\left(W_{t}-c_{t}\right), \quad$ for all $t$.

The isoperimetric condition then is:

$$
\text { (D15) } \quad \lim _{s \rightarrow \infty}\left(\prod_{s=t}^{T} R_{s}^{-1} W_{s}\right) \rightarrow 0
$$

Note that, only if this condition is imposed, we can solve the first-order difference equation in (D14) as:

$$
\text { (D15) } \quad c_{t}+\left(c_{t+1} / R_{t+1}\right)+\left(c_{t+2} / R_{t+1} R_{t+2}\right)+\ldots=W_{t} .
$$

That is the present value of consumption equals initial wealth. If (D15) does not hold, it is possible to draw down wealth to become infinitely negative as time goes to infinity and thus consume extra.

## E. CONDITIONAL Expectations and THE LAW OF ITERATED Expectations

Expectations must always be conditioned on a particular information set. Often this information set includes only the basic characterization of a distribution or a model; we then take unconditional expectations. Essentially expectations are taken without use of specific information. If specific relevant information is available, then mathematical expectations are taken conditional on the information.

Here we prove the Law of Iterated Expectations. This law states essentially that in taking expectations of expectations, the expectation conditional on the least information prevails. Taking expectations of expectations is often important in economic models. For instance, if one wants to forecast the actions of others, or if an average of various conditional expectations must be obtained.

For simplicity we shall first proved the Law of Iterated Expectations for the case when unconditional and conditional expectations are combined. Consider forecasting (forming expectations about) variable $X$ with and without information concerning a correlated variable $Y$. Then, the Law says:
(E1a) $\quad E[E(X \mid Y)]=E(X)$, and
(E1b) $\quad E[E(X) \mid Y]=E(X)$.

In words: the unconditional expectation of the conditional expectation of $X$ is equal to the unconditional expectation of $X$, and the conditional expectation of the unconditional expectation of $X$ is (also) equal to the unconditional expectation of $X$. If you guess what someone with less information than you would do, you base your guess on the lesser information that the other has -- there is no point in using your own larger information set; if you guess what someone with more information would do, your guess must be based on your own, smaller, information set.

To prove (E1) use conditional expectations and Bayes' Law. For (E1a):

$$
\begin{equation*}
E[E(X \mid Y)]=E\left[\int_{x \in X} x f_{x \mid y}(x \mid y) d x\right]=\int_{y \in Y} \int_{x \in X} x f_{x \mid y}(x \mid y) d x f_{y}(y) d y \tag{E2}
\end{equation*}
$$

where $f_{i}()$ indicates a marginal, conditional, or a joint density, with the type and variable revealed by its subscript. By Bayes' Law, $f_{x \mid y}(x \mid y) f_{y}(y)=f_{x y}(x, y)$; hence:

$$
\begin{equation*}
\int_{y \in Y} \int_{x \in X} x f_{x \mid y}(x \mid y) d x f_{y}(y) d y=\int_{y \in Y} \int_{x \in X} x f_{x y}(x, y) d x d y . \tag{E3}
\end{equation*}
$$

Changing the order of integration and realizing that $\int_{y \in Y} f_{x y}(x, y) d y=f_{x}(x)$ :

$$
\begin{equation*}
E[E(X \mid Y)]=\int_{x \in X} x f_{x}(x) d x=E(X) \tag{E4}
\end{equation*}
$$

To prove (E1b) consider that:

$$
\begin{array}{r}
E[E(X) \mid Y]=E\left[\int_{x \in X} x f_{x}(x) d x \mid Y\right]=\int_{y \in Y x \in X} \int_{x \in X} x f_{x}(x) d x f_{y}(y) d y  \tag{E5}\\
=\int_{x \in X} x f_{x}(x) d x \int_{y \in Y} f_{y}(y) d y=\int_{x \in X} x f_{x}(x) d x=E(X)
\end{array}
$$

It is easy to prove the more general result now, that:
(E6a) $\quad E[E(X \mid Y, Z) \mid Y]=E(X \mid Y)$, and
(E6b) $\quad E[E(X \mid Y) \mid Y, Z]=E(X \mid Y)$.

Again, the smaller information set dominates. The proof is omitted here since it is very similar to the above proof of the simpler case, but using more complex versions of Bayes's Law such as $f_{x \mid y z}(x \mid y, z) f_{y \mid z}(y \mid z)=f_{x y \mid z}(x, y \mid z)$.

