## Chapter IX. Specific Dynamic Asset Pricing Models

The previous chapter examined properties of dynamic asset pricing in general. While these models yield general insights, they are not amenable to empirical testing. The reason for the Merton ICAPM model is that the set of variable that may serve as proxies for changes in investment opportunities is too broad. Essentially, any variable that forecasts future returns would be priced in the Merton model. On the other hand, the Breeden CCAPM provides a factor model with aggregate consumption as the only possible factor. The limitation here is that consumption is very difficult to observe. In particular, spending on durable consumption items is not likely to be closely related to durable goods consumption. If durable consumption is excluded, the remaining consumption in the form of spending on nondurables is just too smooth. It is hard to believe that in today's (developed) economies economic factors seriously restrict nondurable consumption, consisting mostly of necessities. The objective in this chapter is to provide some specific models and theories that provide clear and testable implications.

## 1. Production-Based Asset Pricing

## (a) The Lucas Asset Pricing Model

The Lucas (1978) model is based on a simple endowment economy. A representative investor maximizes expected lifetime utility

$$
\begin{equation*}
E_{0} \sum_{t=0}^{\infty} \beta^{t} u\left(c_{t}\right), \quad 0<\beta<1 \tag{1}
\end{equation*}
$$

subject to a wealth constraint:

$$
\begin{equation*}
\sum_{i=1}^{n} p_{t}^{i} s_{t+1}^{i}+q_{t} b_{t+1}=\sum_{i=1}^{n}\left(p_{t}^{i}+d_{t}^{i}\right) s_{t}^{i}+b_{t}-c_{t} \tag{2}
\end{equation*}
$$

Here $p_{t}{ }^{i}$ indicates the price of a share in asset $i, s_{t}{ }^{i}$ indicates the number of shares held at the beginning of period $t$ (and thus is not similar to the use of that symbol in previous chapters where it refers to portfolio share), $q_{t}$ the price of a discount bond, and $b_{t}$ the number of discount bonds held.

The assets should be thought of as "trees" bearing dividends in the form of "fruit". No other means of production exists and the fruit is perishable so that it can not be stored and consumed in future periods. Defining returns as before, $R_{t+1}^{i}=\left(d_{t+1}^{i}+p_{t+1}^{i}\right) / p_{t}^{i}$, the first-order conditions for the risky assets and the riskless asset can be shown to yield:

$$
\begin{align*}
& E_{t}\left[\left(R_{t+1}^{i}-R_{t+1}^{f}\right) u_{c}\left(c_{t+1}\right)\right]=0, \quad \text { for all } i .  \tag{3}\\
& u_{c}\left(c_{t}\right)=\beta E_{t}\left[R_{t+1}^{i} u_{c}\left(c_{t+1}\right)\right], \quad \text { for all } i . \tag{4}
\end{align*}
$$

## Section 1. Production-Based Asset Pricing

We will focus first on equation (4) for an asset representing the market portfolio. This asset should be worth the sum of the prices of all assets and generate a dividend that equals the dividend for the whole market.

In equilibrium, the demands for all shares should equal their supplies. But, given the interpretation that share $\gamma$ represents ownership to a $\gamma$ share of the dividends, the supply of shares for any asset must equal one. If there is no riskless "fruit" technology, then the riskless asset must be in zero net supply under the presumption that, in principle, borrowing and lending may occur at the same rate. Thus, in equilibrium:

$$
\begin{equation*}
s_{t}^{i}=1, \quad b_{t}=0, \quad \text { for all } i \text { and } t \tag{5}
\end{equation*}
$$

Substituting the equilibrium conditions into equation (2) produces:

$$
\begin{equation*}
c_{t}=\sum_{i=1}^{n} d_{t}^{i}=y_{t} \tag{6}
\end{equation*}
$$

The second equality follows since dividends are the only form of production and the only source of income in this endowment economy. Equation (6), of course, must hold given equation (5) due to Walras' Law - the goods market must clear automatically once all other markets clear.

Now apply equation (4) to the market asset:

$$
\begin{equation*}
u_{c}\left(y_{t}\right)=\beta E_{t}\left[R_{t+1} u_{c}\left(y_{t+1}\right)\right] \tag{7}
\end{equation*}
$$

where $R_{t+1} \equiv\left(p_{t+1}+y_{t+1}\right) / p_{t}$, with $\sum_{i=1}^{n} p_{t}{ }^{i} \equiv p_{t}$.
The price of the market asset can thus be found by solving the first order difference equation (7):

$$
\begin{equation*}
p_{t}=\sum_{j=1}^{\infty} \beta^{j} E_{t}\left[y_{t+j} u_{c}\left(y_{t+j}\right)\right] / u_{c}\left(y_{t}\right) . \tag{8}
\end{equation*}
$$

Stock market values thus depend on current and future aggregate production levels. A simpler expression for stock values is hard to obtain except in special cases. For $\log$ utility we obtain directly from equation (8) that $p_{t}=\beta y_{t} /(1-\beta)$, so that $R_{t+1}=(1 / \beta) y_{t+1} / y_{t}$. That is, the stock market price is proportional to current aggregate production but does not depend on future production. This is a consequence of the fact that logarithmic preferences cause individuals to be myopic. The market return can be motivated based on the idea of intertemporal substitution: when $y_{t}$ is low, investors want to sell trees in order to consume more now.

To obtain more insight into this issue and the effect of future output on stock market values, consider the effect of a change in $y_{t+j}$ on $p_{t}$ :

$$
\begin{equation*}
\operatorname{sgn} \partial p_{t} / \partial y_{t+j}=\operatorname{sgn} E_{t}\left[u_{c}\left(y_{t+j}\right)+u_{c c}\left(y_{t+j}\right) y_{t+j}\right] \tag{9}
\end{equation*}
$$

If aggregate production changes are persistent, then further effects will occur in subsequent periods, but all will have the same sign. Sufficient conditions for a positive (negative) sign then are that the coefficient of relative risk aversion is
everywhere less than (larger than) one:

$$
\begin{align*}
A_{t+j}^{R}\left(y_{t+j}\right) & \equiv-\left[u_{c c}\left(y_{t+j}\right) y_{t+j}\right] / u_{c}\left(y_{t+j}\right) \stackrel{<}{<} 1 \text { for all } \mathrm{y}_{\mathrm{t}+\mathrm{j}}  \tag{10}\\
& \rightarrow \operatorname{sgn} \partial p_{t} / \partial y_{t+j} \stackrel{>}{(<)} 0
\end{align*}
$$

The $\log$ case is of course the case where the coefficient of relative risk aversion is exactly equal to one. If the coefficient of risk aversion is less than one, the value of the future dividend income $y_{t+j} u_{c}\left(y_{t+j}\right)$, as evaluated at the marginal benefit of additional current consumption, rises in $y_{t+j}$. There are two effects: a straightforward income effect plus a substitution effect related to the fact that dividends are less valuable when aggregate consumption is high with associated low marginal utility of consumption. The (intertemporal) substitution effect is bigger when the utility function is "more concave." As a result, with higher risk aversion the sign flips. More precisely, if the coefficient of risk aversion is larger than one, the value of the future dividend income and thus the stock market value rises in future output $y_{t+j}$. This result may explain the puzzling phenomenon that positive news on the growth of future aggregate output tends to lower stock market values. In practice, of course, we will observe something like: "The news of continuing high growth in GDP has fueled speculation of interest rate increases by the Fed. As a result, stock prices tumbled." A higher interest rate is similar to an increase in the discount rate for future dividends.

It is straightforward, based on equations (3), (4), and (6), and assuming normality, to derive an asset pricing equation for each asset analogously to the derivation of the CCAPM:

$$
\begin{align*}
& \mu_{t+1}^{i}-r_{t}^{f}=\beta_{i y}\left(\mu_{t+1}^{y}-r_{t}^{f}\right), \text { with }  \tag{11}\\
& \beta_{i y}=\operatorname{Cov}_{t}\left(r_{t+1}^{y}, r_{t+1}^{i}\right) / \operatorname{Var}_{t}\left(r_{t+1}^{y}\right) .
\end{align*}
$$

Here $r_{t+1}^{y}$ may represent either the return on an asset perfectly correlated with aggregate production or the growth rate of aggregate production itself.

## (b) The Brock Model

The Lucas "fruit tree" model is a general equilibrium model in the limited sense that it considers the joint equilibrium implications of all decisions made in the economy. However, a key variable, production, is exogenous. Thus, asset prices and returns are explained for exogenously given equilibrium marginal utilities of consumption as in equation (7). Another standard model, the life-cycle model of consumption, takes market returns as given exogenously and explains how marginal utilities of consumption and consumption itself change over time. Below we consider a third type of model in which both returns as well as consumption and production levels are explained endogenously, the Brock model.

Consider a complete markets economy based on Brock (1982) but using the specific example developed in Balvers, Cosimano, and McDonald (1990). A representative firm maximizes the expected present value of the dividends paid to the stockholders. Investment is assumed to lead to capital with a one period lag but depreciates fully in one
period of use, that is $i_{t}=k_{t+1}$. Production $y_{t}$ depends on capital with a random productivity shock $\theta_{t}$ that is serially uncorrelated. Thus the firm maximizes:
(12) $\max \quad E_{t} \sum_{j=0}^{\infty} m_{t, t+j} d_{t+j}$

Subject to:

$$
\begin{align*}
& d_{t}=y_{t}-k_{t+1}  \tag{13}\\
& y_{t}=\theta_{t} A e^{\delta t} k_{t}^{\alpha}, \tag{14}
\end{align*}
$$

where $m_{t, t+j}$ is the stochastic discount factor for each period $t+j$, based on the starting point at time $t ; \quad m_{t, t}=1$. Production includes a standard exponential time trend.

The decision problem based on equations (12) - (14) becomes:

$$
\begin{equation*}
V\left(k_{t}\right)=\max _{k_{t+1}}\left(y_{t}-k_{t+1}+E_{t}\left[m_{t, t+1} V\left(k_{t+1}\right)\right]\right) \tag{15}
\end{equation*}
$$

subject to equation (14). The first-order condition and the envelope condition become:

$$
\begin{align*}
& 1=E_{t}\left[m_{t, t+1} V_{k}\left(k_{t+1}\right)\right]  \tag{16}\\
& V_{k}\left(k_{t}\right)=\alpha y_{t} / k_{t} \tag{17}
\end{align*}
$$

Combining the envelope and first-order condition yields:

$$
\begin{equation*}
1=\alpha E_{t}\left[m_{t, t+1} y_{t+1} / k_{t+1}\right] \tag{18}
\end{equation*}
$$

The consumer's decision problem is very similar to the decision problem in the Lucas model. First, goods market equilibrium implies that $c_{t}=d_{t}$. Second, assuming a log utility function implies, following the same steps as in the Lucas model, that:

$$
\begin{equation*}
m_{t, t+1}=\beta u_{c}\left(c_{t+1}\right) / u_{c}\left(c_{t}\right) \rightarrow m_{t, t+1}=\beta d_{t} / d_{t+1} \tag{19}
\end{equation*}
$$

The first equation follows by considering the price of an asset at time $t$ that yields a dividend of one unit at time $t+l$ and zero at all other times, plus comparing equations (8) and (12). Substituting equation (19) into equation (18) yields:

$$
\begin{equation*}
1=\alpha \beta E_{t}\left[\left(d_{t} / d_{t+1}\right)\left(y_{t+1} / k_{t+1}\right)\right] \tag{20}
\end{equation*}
$$

Next step is to guess a solution for the value function or for the decision rule. In this case assume:
(21) $k_{t+1}=F y_{t}$,
where $F$ is an undetermined coefficient. Equation (21) implies from equation (13) that:

$$
\begin{equation*}
d_{t}=(1-F) y_{t} . \tag{22}
\end{equation*}
$$

Substitute both equations (21) and (22) into equation (20) to find $F=\alpha \beta$, so that:

$$
\begin{equation*}
k_{t+1}=\alpha \beta y_{t}, \quad c_{t}=d_{t}=(1-\alpha \beta) y_{t} . \tag{23}
\end{equation*}
$$

Aggregate production from equation (14) then becomes:

$$
\begin{equation*}
y_{t+1}=\theta_{t+1} A e^{\delta(t+1)}(\alpha \beta)^{\alpha} y_{t}^{\alpha} \tag{24}
\end{equation*}
$$

Lastly, since from the consumer decision problem we have $R_{t+1}=(1 / \beta) d_{t+1} / d_{t}$, we find from equation (23) that:

$$
\begin{equation*}
R_{t+1}=(1 / \beta) y_{t+1} / y_{t} . \tag{25}
\end{equation*}
$$

Thus, aggregate production shocks are persistent as follows from equation (24). A positive technology shock (increase in $\theta_{t}$ ) raises aggregate production which is spent in part on investment. As a result the capital stock for the next period is larger which leads to more production in the next period (all else equal). The reason that higher aggregate production leads to more investment is that, if all were spent on consumption, the marginal utility of consumption would go down too far, so that investment, leading to future consumption, is a more profitable alternative. Stock returns vary to allow this process, which leads to a smoother consumption sequence, to occur. If current output is low, consumption will be low and the marginal utility of consumption high. Returns for the upcoming period now are expected to be high as shown in equation (23). The reason is that in order to generate investment, the returns on investing would have to be high to entice consumers to investment in a state where marginal utility of consumption is high. In other words, if current output is lower, the savings schedule shifts to the left, causing higher returns and lower aggregate investment and savings. See Figure 1.

## (c) Predictability of Returns

Taking expectations in equation (25) gives:

$$
\begin{equation*}
E_{t} R_{t+1}=(1 / \beta) E_{t} y_{t+1} / y_{t} \tag{26}
\end{equation*}
$$



## Stock Returns over the Business Cycle

The effect of a lower level of business activity (such as industrial production) on the stock return if business activity is mean reverting. Lower current output and income implies that households are less willing to transfer their wealth to future time periods which are presumed to be more plentiful.

Market returns can be predicted to the extent that aggregate output is forecastable. The intuitive reason is that a business cycle upswing, in the sense that aggregate production for the next period is expected to increase, leads to a desire to save less currently. This is implied by the life cycle hypothesis. Hence, the demand for stocks falls, raising expected stock returns for the upcoming period. No arbitrage opportunities arise since the same is true for all stocks. In fact, the stochastic discount factor is changing over time which causes the predictability in returns. But as long as the proper stochastic discount factor is used, arbitrage is by definition ruled out.

Balvers, Cosimano, and McDonald (1990) test the implication of the Brock model that stock returns can be forecast by forecasting aggregate production. Taking logs in equations (23) and (24) produces:

$$
\begin{align*}
& \ln y_{t+1}=\gamma+\delta t+\alpha \ln y_{t}+\varepsilon_{t+1}  \tag{27}\\
& r_{t+1}=-\ln \beta+\ln y_{t+1}-\ln y_{t} \tag{28}
\end{align*}
$$

where the variable definitions are obvious based on equations (23) and (24). Note that the empirical implications of this Brock model are quite similar to the implications of the Lucas model. Both equations (27) and (28) are consistent with the Lucas model. The only difference is that the Brock model explicitly implies an equation like (27) whereas in the Lucas model the stochastic process for aggregate output can be anything. Of course, the similarity to the Lucas model also betrays the fact that we are in fact estimating the CCAPM implications for the market return with aggregate consumption replaced by the less direct but more accurately measured aggregate production variable.

Combining equations (27) and (28) yields that:

$$
\begin{equation*}
r_{t+1}=(\gamma-\ln \beta)+\delta t+(\alpha-1) \ln y_{t}+\varepsilon_{t+1} \tag{29}
\end{equation*}
$$

## Chapter VII. Specific Dynamic Asset Pricing Models

In a regression framework, the right-hand side variables pre-date the return on the left-hand side. The prediction is that current aggregate production has a negative impact on the market return for the upcoming period. This follows because of "trend reversion" in aggregate production - equation (27) implies that, say, positive aggregate output shocks slowly wear out, causing aggregate output to return to trend. As a result, anticipated growth in aggregate output would be unusually low in response to a positive shock, causing expected market returns to be low as well (since households desire to save more as a means of smoothing consumption).

The empirical results in Balvers, Cosimano, and McDonald (1990), using annual data for the value-weighted CRSP stock index for the period 1947-1988 and industrial production to proxy for aggregate production are as follows: (a) the coefficient for the log of industrial production is significantly negative; (b) the $\mathrm{R}^{2}$ is $22 \%$ so that there appears to be significant forecastability in market returns; (c) if the expected growth rate is estimated first based on equation (27) and is then used in equation (28), results are virtually identical to the results of estimating equation (29). This suggests that the effect of aggregate production on returns occurs indeed as predicted from equation (28) rather than through some other unknown mechanism. Further, (d) the regression $\mathrm{R}^{2}$ rises to $50 \%$ if the horizon is extended from one to five years; and (e) the dividend yield variable (that was shown in earlier work by Fama and French to be important in forecasting stock returns) becomes insignificant once aggregate production is included in the forecasting regression.

The empirical work thus has focused on the time-series implications of the production-based asset pricing model (PCAPM). The results in effect support the CCAPM, together with the presumption that aggregate production is more accurately measured than aggregate consumption. The cross-sectional implications of the PCAPM have not been tested explicitly. While empirical work has found that GDP growth is a factor in explaining cross-section return differences, there has been no attempt at a more theory-based test directly from, say, equation (11) and using an approach such as in Mankiw and Shapiro (1985).

## 2. Investment-Based Asset Pricing

Cochrane (1991) introduced a different perspective on asset pricing by exploiting the link between returns on physical investment and the returns on the equity asset that lays claim on the returns from physical investment. At times the ensuing literature is termed production-based asset pricing but we will use Cochrane's (1996) term Investment-Based Asset Pricing.

## (a) Stock Returns and Physical Investment

Restoy and Rockinger (1994) provide a nice theoretical derivation of the link between stock returns and investment in a model with adjustment costs. The model is closely related to the $Q$-theory of investment but with the important difference that the stochastic discount factor is allowed to vary over time, whereas the $Q$-theory models typically assume a constant risk-free real interest rate as the discount factor.

In the absence of arbitrage opportunities we know that the value of any firm can be given as:

$$
\begin{equation*}
V_{t}=d_{t}+p_{t}=E_{t} \sum_{j=0}^{\infty} m_{t, t+j} d_{t+j} . \tag{1}
\end{equation*}
$$

## Section 1. Production-Based Asset Pricing

The value of the firm $V$ is equal to current dividends $d$ plus the ex-dividend value of the firm $p$. Assume that technology shocks $\theta_{t}$ follow a Markov process and that capital $k_{t}$ does not fully depreciate in one period. Further assuming that the firm's choice variables are labor inputs $L_{t}$ and physical investment $I_{t}$, the Bellman equation can be written as:

$$
\begin{equation*}
V\left(k_{t}, \theta_{t}\right)=\max _{I_{t}, L_{t}}\left(d_{t}+E_{t}\left[m_{t, t+1} V\left(k_{t+1}\right), \theta_{t+1}\right]\right), \tag{2}
\end{equation*}
$$

where:

$$
\begin{align*}
& d_{t}=\theta_{t} f\left(k_{t}, L_{t}\right)-w_{t} L_{t}-I_{t},  \tag{3}\\
& k_{t+1}=g\left(k_{t}, I_{t}\right) . \tag{4}
\end{align*}
$$

The functions $f()$, and $g()$ are assumed to be homogeneous of degree one. Dividends paid to equity holders are equal to production revenue $\theta_{t} f\left(k_{t}, L_{t}\right)$ minus labor costs $w_{t} L_{t}$ and investment. Next period's capital stock depends (positively) on the current capital stock and current investment. Due to installation costs the relation between current investment and the next-period capital stock is not necessarily proportional. Note that this firm is not leveraged so that all net revenues go to the equity holders. A more detailed model of this type, which also includes retained earnings and bonds is given in Altug and Labadie (1994, pp.165-168).

The first-order conditions for the firm are:
(6) $1=g_{I}\left(k_{t}, I_{t}\right) E_{t}\left[m_{t, t+1} V_{k}\left(k_{t+1}, \theta_{t+1}\right)\right]$.

The envelope condition produces:

$$
\begin{equation*}
V_{k}\left(k_{t}, \theta_{t}\right)=\theta_{t} f_{k}\left(k_{t}, L_{t}\right)+g_{k}\left(k_{t}, I_{t}\right) E_{t}\left[m_{t, t+1} V_{k}\left(k_{t+1}, \theta_{t+1}\right)\right] . \tag{7}
\end{equation*}
$$

Combining equations (6) and (7) yields;

$$
\begin{equation*}
V_{k}\left(k_{t}, \theta_{t}\right)=\theta_{t} f_{k}\left(k_{t}, L_{t}\right)+\left[g_{k}\left(k_{t}, I_{t}\right) / g_{I}\left(k_{t}, I_{t}\right)\right] \tag{8}
\end{equation*}
$$

Updating equation (8) by one period and substituting into equation (7) gives:

$$
\begin{equation*}
1=g_{I}\left(k_{t}, I_{t}\right) E_{t}\left(m_{t, t+1}\left\{\theta_{t} f_{k}\left(k_{t}, L_{t}\right)+\left[g_{k}\left(k_{t+1}, I_{t+1}\right) / g_{I}\left(k_{t+1}, I_{t+1}\right)\right]\right\}\right) \tag{9}
\end{equation*}
$$

The rest of the derivation of stock returns is necessary to show that the part of the right-hand side of equation (9) that multiplies the stochastic discount factor is equal to the stock return. The basic reason that this turns out to be the case is due to the homogeneity assumptions that guarantee, as we kind of know from Hayashi (1982), that marginal
$Q$ equals average $Q$ or, similarly, that $V_{k}=k V$.
We know that $p_{t}$ can be written as:

$$
\begin{equation*}
p_{t}=E_{t}\left[m_{t, t+1}\left(p_{t+1}+d_{t+1}\right)\right] \tag{10}
\end{equation*}
$$

Using equation (3), the homogeneity of $f($ ), and equation (5) yields:

$$
\begin{equation*}
p_{t}=E_{t}\left\{m_{t, t+1}\left[p_{t+1}+\theta_{t+1} f_{k}\left(k_{t+1}, L_{t+1}\right) k_{t+1}-I_{t+1}\right]\right\} \tag{11}
\end{equation*}
$$

The homogeneity of $g()$ implies that $k_{t+1}=g_{k}\left(k_{t}, I_{t}\right) k_{t}+g_{I}\left(k_{t}, I_{t}\right) I_{t}$. Updating this equation by one period to eliminate investment in equation (11) produces:

$$
p_{t}=E_{t}\left(m_{t, t+1}\left\{p_{t+1}+\left[\theta_{t+1} f_{k}\left(k_{t+1}, L_{t+1}\right)+\frac{g_{k}\left(k_{t+1}, I_{t+1}\right)}{g_{I}\left(k_{t+1}, I_{t+1}\right)}\right] k_{t+1}-\frac{k_{t+2}}{g_{I}\left(k_{t+1}, I_{t+1}\right)}\right\}\right)
$$

Now use equation (9) in the above equation to simplify the middle term. This gives:
(12) $p_{t}-\frac{k_{t+1}}{g_{I}\left(k_{t}, I_{t}\right)}=E_{t}\left[m_{t, t+1}\left(p_{t+1}-\frac{k_{t+2}}{g_{I}\left(k_{t+1}, I_{t+1}\right)}\right)\right]$.

Equation (12) can be written as $x_{t}=E_{t}\left(m_{t, t+1} x_{t+1}\right)$ with $x_{t}=p_{t}-\left[k_{t+1} / g_{I}\left(k_{t}, I_{t}\right)\right]$.
The solution of this first-order difference equation (ruling out bubbles) is simply that $x_{t}=0$.
Thus:
(13) $p_{t}=\frac{k_{t+1}}{g_{I}\left(k_{t}, I_{t}\right)}$.

Without adjustment costs we would of course have $p_{t}=k_{t+1}$ which implies $Q=1$.
The gross return on equity can now be written as:

$$
\begin{equation*}
R_{t+1}=\frac{\left[k_{t+2} / g_{I}\left(k_{t+1}, I_{t+1}\right)\right]+d_{t+1}}{k_{t+1} / g_{I}\left(k_{t}, I_{t}\right)} \tag{14}
\end{equation*}
$$

Employing again the homogeneity of $f()$ and $g()$ and equation (5) yields:

$$
\begin{equation*}
R_{t+1}=\frac{\left[g_{k}\left(k_{t+1}, I_{t+1}\right) k_{t+1} / g_{I}\left(k_{t+1}, I_{t+1}\right)\right]+\theta_{t+1} f_{k}\left(k_{t+1}, L_{t+1}\right) k_{t+1}}{k_{t+1} / g_{I}\left(k_{t}, I_{t}\right)} \tag{15}
\end{equation*}
$$

Canceling the $k_{t+1}$ term and adding superscripts $i$ to indicate firm-specific differences provides a non-standard asset pricing equation:

$$
\begin{equation*}
R_{t+1}^{i}=\left(\theta_{t+1}^{i} f_{k}^{i}\left(k_{t+1}^{i}, L_{t+1}^{i}\right)+\frac{g_{k}^{i}\left(k_{t+1}^{i}, I_{t+1}^{i}\right)}{g_{I}^{i}\left(k_{t+1}^{i}, I_{t+1}^{i}\right)}\right) g_{I}^{i}\left(k_{t}^{i}, I_{t}^{i}\right) . \tag{16}
\end{equation*}
$$

Cochrane (1991) obtained equation (16) in the more specific context of a complete markets economy. His intuitive derivation of this result is as follows. The right-hand side of equation (16) represents the physical investment return of firm $i$. It is obtained from a within-firm type of arbitrage: invest in the current period and then withdraw enough investment in the next period to keep the capital stock for future periods equal to what it would have been without the current period investment; the net payoff per unit extra investment in the current period is the investment return. It is equal to the output gain for period $t+1$ per unit of extra investment: $\left[\theta_{t+1} f_{k}(t+1)\right] g_{I}(t)$ - the marginal effect of investment on the capital stock times the marginal effect of the capital stock increase in production; plus the gain due to reduction in period $t+l$ investment that can occur (to return capital to its original level) because capital has increased: $\left[g_{k}(t+1) / g_{I}(t+1)\right] g_{I}(t)$ - the marginal effect of investment on capital times the marginal effect of period $t+l$ capital on period $t+2$ capital divided by how much investment can be reduced in period $t+l$ to keep capital unchanged.

Some specific functional forms for the $f()$ and $g()$ functions provide a more concrete and operational asset pricing equation. Cochrane (1991) used the following functional forms:

$$
\begin{align*}
& f\left(k_{t}, L_{t}\right)=m p k_{t} k_{t}+m p l_{t} L_{t},  \tag{17}\\
& g\left(k_{t}, I_{t}\right)=(1-\delta)\left\{k_{t}+\left[1-(\gamma / 2)\left(I_{t} / k_{t}\right)^{2}\right] I_{t}\right\}, \tag{18}
\end{align*}
$$

where $m p k_{t}$ and $m p l_{t}$ are time-varying constants. Accordingly, equation (16) becomes:

$$
R_{t+1}^{i}=(1-\delta)\left(\theta_{t+1}^{i} m p k_{t+1}^{i}+\frac{1+\gamma\left(I_{t+1}^{i} / k_{t+1}^{i}\right)^{3}}{1-(3 / 2) \gamma\left(I_{t+1}^{i} / k_{t+1}^{i}\right)^{2}}\right)\left[1-(3 / 2) \gamma\left(I_{t}^{i} / k_{t}^{i}\right)^{2}\right] .
$$

Thus, aside from firm-specific productivity measure and a marginal product of capital measure, firm-specific investment-to-capital ratios for two periods are necessary to explain the cross-section of returns.

A simpler assumption for the $f()$ and $g()$ functions is:

$$
\begin{equation*}
g\left(k_{t}, I_{t}\right)=(1-\delta) k_{t}^{\omega} I_{t}^{1-\omega}, f\left(k_{t}, L_{t}\right)=A k_{t}^{\alpha} L_{t}^{1-\alpha} . \tag{19}
\end{equation*}
$$

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This functional form for the investment installation function makes sense since it implies homogeneity of degree one, a depreciation rate of $\delta$, a positive impact of the existing capital stock and a positive impact of current investment on the future capital stock. The return then becomes:

$$
R_{t+1}^{i}=\left[(1-\omega) \alpha\left(y_{t+1}^{i} / y_{t}^{i}\right)\left(y_{t}^{i} / I_{t}^{i}\right)\right]+\left[\omega\left(I_{t+1}^{i} / I_{t}^{i}\right)\right] .
$$

Thus stock returns are weighted averages of output growth (with the output to investment ratio as part of the weight) and investment growth.

## (b) Empirical Tests of Investment-Based Asset Pricing

Aside from the aforementioned papers by Cochrane (1991) and Restoy and Rockinger (1994), not much attention has been paid to Investment-Based Asset Pricing. Basu and Vinod (1994) extend the work of Balvers, Cosimano and McDonald (1990) and Cochrane (1991) to look at the effects of the degree of economies of scale on asset pricing. Arroyo (1996) assumes a constant rate for discounting firm profits but extends Cochrane (1991) to consider differential costs of financing. All of these papers in their empirical work look only at the risk free asset and the market asset.

Braun (1991) considers the cross-sectional implications of the investment-based approach by, basically, estimating equation (16) directly. Cochrane (1996) also contemplates the cross-sectional implications of the investmentbased approach. He tests an equation similar to equation (16) above, but derived from a convex adjustment cost specification rather than the "costly transformation of investment to capital" approach employed by Cochrane (1991) and Restoy and Rockinger (1994). Since marginal production costs are assumed constant and there are no technology shocks, the regression, essentially includes the current and lagged investment-to-capital ratios as explanatory variables only. Cochrane then examines the stock returns of the ten deciles based on size from CRSP plus the 3-month T-Bill. He employs a Generalized Method of Moments (GMM) approach that does not directly incorporate the cross-sectional variation of investment-to-capital ratios but uses dividend-to-price ratios and the term premium as instruments. The complexity of the method makes it difficult to provide a clear assessment of the results, but they appear to be competitive with other less theoretical approaches.

Theoretically, at least, direct estimation of an equation like equation (16) involves asset returns that should differ by firm-specific investment-to-capital ratio. An approach that is theoretically quite different would insist that, in complete markets, the MRIS = MRIT (marginal rate of intertemporal substitution equals marginal rate of intertemporal transformation). The stochastic discount factor can then directly be set equal to the MRIT which depends on aggregate production factors rather than the MRIS used in the CCAPM which depends on aggregate consumption. As production is presumably measured with less error than consumption, the production-based approach would be expected to perform better. The factors would then be related to technology shocks, the capital stock, the labor stock, and other production factors and could be obtained directly by employing the generalized Stein's Lemma if normality of the returns is assumed. As such, the asset pricing model may provide evidence for or against Real Business Cycle theory. At this time, such an approach has not been attempted, neither for U.S. stock returns nor for stock returns across countries.

## 3. The Conditional CAPM

An implicit assumption of the CAPM is that the market beta is constant over time. In the derivation of the standard CAPM this assumption does not enter as there is only one period. However, in an intertemporal context a specific assumption about the time series properties of the market beta must be made. The conditional CAPM looks at the implications of allowing beta to change over time.

## (a) The Premium Beta

Consider an intertemporal CAPM model that allows betas as well as risk free rate and market premium to change over time. This model is based on Jagannathan and Wang (1996). Then we can write in principle:

$$
\begin{equation*}
E_{t-1} r_{t}^{i}=r_{t}^{f}+\beta_{t-1}^{i} E_{t-1} e_{t} \tag{1}
\end{equation*}
$$

where: $E_{t-1} e_{t} \equiv E_{t-1} r_{t}^{m}-r_{t}^{f}, \quad \beta_{t-1}^{i}=\operatorname{Cov}_{t-1}\left(r_{t}^{i}, r_{t}^{m}\right) / \operatorname{Var}_{t-1}\left(r_{t}^{m}\right)$

As Jagannathan and Wang point out, based on Merton's ICAPM, other betas may be important if individuals want to hedge against changes in the investment opportunity set. To avoid this complication, assume here that the market premium of hedging betas is zero so that these can be ignored.

Ideally, asset pricing should take into account the information about the beta and market return at each point in time. However, this is difficult and may be practically infeasible. If we take an easier approach and take unconditional expectations in equation (1), we obtain:

$$
\begin{equation*}
E r_{t}^{i}=r_{t}^{f}+E \beta_{t-1}^{i} E e_{t}+\operatorname{Cov}\left(E_{t-1} e_{t}, \beta_{t-1}\right) \tag{2}
\end{equation*}
$$

where we used the definition of covariance to obtain the covariance term in equation (2) and also employed the Law of Iterated Expectations as described in Appendix E. There is a "standard" effect of the (expected) beta on expected return, but in addition there is an effect related to the covariance of beta with the expected market return. This second effect enters because assets that have higher betas when the expected market risk premium is higher should have higher expected returns on average. Note that for the sake of simplicity we assume that a true risk free rate exists so that this rate moves perfectly predictably over time even though it need not be constant. The exposition in Jagannathan and Wang does not make this assumption.

The conditional beta can be decomposed into three parts by regressing it on the difference between conditional and unconditional market excess return:

$$
\begin{equation*}
\beta_{t-1}^{i}=E \beta_{t-1}^{i}+\gamma^{i}\left(E_{t-1} e_{t}-E e_{t}\right)+\eta_{t-1}^{i} \tag{3}
\end{equation*}
$$

The parts thus are: (a) the unconditional mean beta; (b) the part that is correlated with the market excess return; and (c) a part that has mean of zero and is uncorrelated with the market excess return, as follows from the properties of regression. It follows by using the regression property that $\gamma^{i}=\operatorname{Cov}\left(\beta_{t-1}^{i}, E_{t-1} e_{t}\right) / \operatorname{Var}\left(E_{t-1} e_{t}\right)$ in equation (2) that:

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$$
\begin{equation*}
E r_{t}^{i}=r_{t}^{f}+E \beta_{t-1}^{i} E e_{t}+\gamma^{i} \operatorname{Var}\left(E_{t-1} e_{t}\right) \tag{4}
\end{equation*}
$$

Next define the "residual":

$$
\begin{equation*}
\varepsilon_{t}^{i}=r_{t}^{i}-r_{t}^{f}-\beta_{t-1}^{i}\left(r_{t}^{m}-r_{t}^{f}\right) \tag{5}
\end{equation*}
$$

It follows from equation (1) that:

$$
\begin{equation*}
E_{t-1} \varepsilon_{t}^{i}=E_{t-1}\left(\varepsilon_{t}^{i} r_{t}^{m}\right)=0 \tag{6}
\end{equation*}
$$

Further, obviously, $E_{t-1}\left(\varepsilon_{t}^{i} E_{t-1} r_{t}^{m}\right)=0$. Taking unconditional expectations then yields:

$$
\begin{equation*}
E \varepsilon_{t}^{i}=E\left(\varepsilon_{t}^{i} r_{t}^{m}\right)=E\left(\varepsilon_{t}^{i} E_{t-1} r_{t}^{m}\right)=0 \tag{7}
\end{equation*}
$$

Next use the factorization of beta in equation (3) and combine with equation (5) to produce:

$$
\begin{align*}
& r_{t}^{i}=r_{t}^{f}+E\left(\beta_{t-1}^{i}\right)\left(r_{t}^{m}-r_{t}^{f}\right)+  \tag{8}\\
& \qquad \gamma^{i}\left(E_{t-1} e_{t}-E e_{t}\right)\left(r_{t}^{m}-r_{t}^{f}\right)+\eta_{t-1}^{i}\left(r_{t}^{m}-r_{t}^{f}\right)+\varepsilon_{t}^{i} .
\end{align*}
$$

Employ equation (8) to find the covariances between the return on asset $i$ and the market return as well as between the return on asset $i$ and the conditional market return:

$$
\begin{equation*}
\operatorname{Cov}\left(r_{t}^{i}, r_{t}^{m}\right)=E\left(\beta_{t-1}^{i}\right) \operatorname{Var}\left(r_{t}^{m}\right)+\gamma^{i} \operatorname{Cov}\left[e_{t}\left(E_{t-1} e_{t}-E e_{t}\right), r_{t}^{m}\right] \tag{9}
\end{equation*}
$$

Note that the eta and epsilon terms vanish. The proof (under mild conditions) that the eta term vanishes is rather tedious and is omitted here. See Appendix A in Jagannathan and Wang (1996) for the formal proof. Similarly,

$$
\begin{align*}
& \operatorname{Cov}\left(r_{t}^{i}, E_{t-1} r_{t}^{m}\right)=E\left(\beta_{t-1}^{i}\right) \operatorname{Cov}\left(r_{t}^{m}, E_{t-1} r_{t}^{m}\right)+  \tag{10}\\
& \gamma^{i} \operatorname{Cov}\left[e_{t}\left(E_{t-1} e_{t}-E e_{t}\right), E_{t-1} r_{t}^{m}\right] .
\end{align*}
$$

It is now easy to rewrite equation (4) as a two-beta equation. First define the betas:

$$
\begin{align*}
& \beta^{i}=\operatorname{Cov}\left(r_{t}^{i}, r_{t}^{m}\right) / \operatorname{Var}\left(r_{t}^{m}\right)  \tag{11}\\
& \delta^{i}=\operatorname{Cov}\left(r_{t}^{i}, E_{t-1} r_{t}^{m}\right) / \operatorname{Var}\left(E_{t-1} r_{t}^{m}\right)
\end{align*}
$$

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Here $\delta^{i}$ is the "premium" beta. Use the beta definitions on the left-hand sides of equations (9) and (10):

$$
\begin{align*}
& \beta^{i}=E\left(\beta_{t-1}^{i}\right) \frac{\operatorname{Var}\left(r_{t}^{m}\right)}{\operatorname{Var}\left(r_{t}^{m}\right)}+\gamma^{i} \frac{\operatorname{Cov}\left[e_{t}\left(E_{t-1} e_{t}-E e_{t}\right), r_{t}^{m}\right]}{\operatorname{Var}\left(r_{t}^{m}\right)},  \tag{12}\\
& \delta^{i}=E\left(\beta_{t-1}^{i}\right) \frac{\operatorname{Cov}\left(r_{t}^{m}, E_{t-1} r_{t}^{m}\right)}{\operatorname{Var}\left(E_{t-1} r_{t}^{m}\right)}+\gamma^{i} \frac{\operatorname{Cov}\left[e_{t}\left(E_{t-1} e_{t}-E e_{t}\right), E_{t-1} r_{t}^{m}\right]}{\operatorname{Var}\left(E_{t-1} r_{t}^{m}\right)}
\end{align*}
$$

Thus, $E\left(\beta_{t-1}^{i}\right)$ and $\gamma^{i}$ can be expressed as a linear function of $\beta^{i}$ and $\delta^{i}$ and the result can then be substituted into equation (4) to yield a two beta formulation.

## (b) Empirical Results

In testing their model Jagannathan and Wang assume somewhat doubtfully that the premium beta can be approximated by the beta between the asset return and the yield spread between low-grade corporate bonds and high grade corporate bonds. Supposedly, the conditional market return moves together with the yield spread pretty closely. The reason provided by Jagannathan and Wang is that the business cycle is best forecast by the yield premium (for which there is some decent support) and that the market risk premium moves closely with the business cycle.

Thus, in effect, Jagannathan and Wang introduce a yield premium beta which has been shown to work in previous studies in explaining the cross-section of U.S. stock returns. Not surprisingly the results are quite strong. They sort CRSP stocks by size and market beta into 100 portfolios following Fama and French (1992) using monthly data from 1963. They obtain an $\mathrm{R}^{2}$ of 0.30 with the market beta insignificant while the premium beta is highly significant. When Jagannathan and Wang also add a beta for human capital to better proxy for the market they obtain an $\mathrm{R}^{2}$ of 0.55 . When the size beta and the book-to-market beta are added (together with the market beta forming the Fama-French three-factor model), the results change little. The interpretation is that the CAPM is saved!

