Chapter VII. General Issues in Valuation and Arbitrage

In this chapter we consider general relationships that exist between discounting cash flows, beta-pricing models and mean-variance efficiency. These relationships are found to exist more generally than in the context of the CAPM only. Much of the discussion in this chapter is my distillation of the material in Cochrane (1999).

1. Stochastic Discount Factors

(a) Complete Markets and the Discount Factor

From equation (1.10) in Chapter V we know that we can write for the value of any asset in a complete markets economy:

(1) $$p_i = E[m(s)\, x_i(s)] \ , \quad \text{where} \quad m(s) = \frac{1}{1 + r^e(s)} > 0 \ .$$

Here $x_i(s)$ represents the asset's payoff in state *s*. The factor $m(s)$ in each state *s* is the inverse of one plus the expected return of the Arrow-Debreu security for state *s* and is thus *positive* in each state. It is also *unique* since only Arrow-Debreu prices are determined uniquely. We can think of the expected return for the Arrow-Debreu security in state *s* as the discount rate for a payoff to be received in state *s*. Accordingly it is natural to think of $m(s)$ as a discount factor that varies stochastically depending on the state. Aside from the term *stochastic discount factor*, $m(s)$, depending on context, is also called the *pricing kernel* (or sometimes *equivalent Martingale measure*, *Radon-Nikodyme derivative*, *marginal rate of intertemporal substitution*, or *benchmark pricing variable*). Its significance is that this one stochastic discount factor prices every asset in the economy. Any asset can be priced by discounting payoffs by the same stochastic discount factor.

Note that we can write equivalently that

(2) $$1 = E[m(s)\, R_i(s)] \ ,$$

where $R_i(s)$ represents the gross return on asset *i* in state *s*.

(b) Examples of Stochastic Discount Factors in Incomplete Market Economies

Markets need not be complete in the context of the CAPM. Under the assumption of normal returns we found in Chapter III (equation 3.12) that the first-order conditions for investor *k* imply that

(3) $$E[u_k^{/}(w_k)(R_i - R_f)] = 0 \ \rightarrow \ m_k = a + b u_k^{/}(w_k) \, .$$

The implication follows by defining *m* consistent with equation (2) and since the first part of equation (3) holds for all *i*. Thus, the stochastic discount factor is related to the marginal utility of wealth and in this incomplete markets context may differ by individual but still prices each asset for that individual.

When we get into dynamic asset pricing theories in the next chapter, we will derive the following standard first-order condition:

$$(4) \qquad \beta\, E_t\{[u_k^{/}(c_t^{\,k})/u_k^{/}(c_{t+1}^{\,k})](R_i - R_f)\} \;=\; 0 \;\;\rightarrow\;\; m_k \;=\; \beta\,[u_k^{/}(c_t^{\,k})/u_k^{/}(c_{t+1}^{\,k})].$$

The stochastic discount factor for individual *k* here equals the marginal rate of intertemporal substitution.

At least in the two cases describe above, it is possible to find stochastic discount factors that prices all assets for a particular individual. To what extent these factors may differ across individuals, whether they most be positive (as is clearly the case in equation 4), whether they are unique for an individual, whether they can be mimicked by a portfolio of marketable assets, and whether a discount factor must exist in all incomplete-market models is not immediately clear and will be considered next.

2. INCOMPLETE MARKETS AND STOCHASTIC DISCOUNT FACTORS

We will prove some quite general results that apply for stochastic discount factors in an incomplete markets context.

(a) Value Additivity

Consider the set of primary assets with imperfectly correlated returns arranged in the column vector $\mathbf{x}$ such that $\mathbf{x}^{\mathbf{T}} = [x_1, x_2, \ldots x_n]$ with asset prices $\mathbf{p}^{\mathbf{T}} = [p_1, p_2, \ldots p_n]$. All primary assets are available to all investors. If we define the *payoff space* X as the set of payoffs available to all assets, then we have $x_i \in X$ for all $i \in \{1, \ldots, n\}$.

Assume now that markets are frictionless in the sense that portfolio combinations of the *n* primary assets are feasible for all individuals at no cost. Or:

> *Assumption 1 (Frictionless Portfolio Formation).* For all i and $j \in \{1, \ldots, n\}$ and for all real numbers *a* and *b*, $x_k \equiv a x_i + b x_j \in X$.

Thus the payoff space consists of all linear combinations of the primary payoffs.

Further assume that *Value Additivity* holds. This implies that the price of a linear combination of payoffs is equal to the linear combination of the prices of the payoffs. Cochrane (1999) calls this assumption the “law of one price” (LOOP) as two alternatives that are essentially equivalent are assumed to have the same price. Cochrane calls this the “happy meal” assumption: the price of a happy meal should just be equal to the price of a burger, fries, and a shake. More in the realm of financial assets, the price of, say, a government bond should be equal to the price of its stripped zero-coupon bond and the prices of the stripped coupons added together. Formally:

Assumption 2 (Value Addditivity). For all real numbers a and b, and for all x_i and $j \in X$, $x_k \equiv a x_i + b x_j \rightarrow p(x_k) = a p(x_i) + b p(x_j)$.

The following general result then applies in an environment where markets need not be complete:

RESULT 1 (VALUE ADDITIVITY AND THE STOCHASTIC DISCOUNT FACTOR). *Provided that Assumption 1 holds:*

$$\textit{Assumption 2} \Leftrightarrow \text{For all } x_i \in \mathrm{X}: \quad E(m^* x_i) = p_i, \quad \text{where } m^* \in \mathrm{X} \text{ and unique within X.}$$

Proof. First, $E(m^* x_i) = p_i \rightarrow$ Value Additivity. Say, $x_k \equiv a x_i + b x_j$. Then it is easy to verify the following $p_k = E(m^* x_k) = E[m^*(a x_i + b x_j) = aE(m^* x_i) + bE(m^* x_k) = a p_i + b p_j$. Second, it is easy to derive that Value Additivity $\rightarrow E(m^* x_i) = p_i$. In vector notation we can write $E(m^* \mathbf{x}) = \mathbf{p}$ and we need to show that a stochastic discount factor exists such that this system of equations holds. Assumption 1 in vector notation implies that X = {$\mathbf{c}^{\mathbf{T}}\mathbf{x}$} where **c** represents any n-vector of real numbers. Assume that $m^* = \mathbf{b}^{\mathbf{T}}\mathbf{x}$ so that it is a member of X. Set $\mathbf{b} = (\mathbf{x}\,\mathbf{x}^{\mathbf{T}})^{-1}\mathbf{p}$. Substituting into $E(m^* \mathbf{x}^{\mathbf{T}}) = \mathbf{p}^{\mathbf{T}}$ shows that the equation holds as needed to be shown. Note that the matrix $\mathbf{x}\,\mathbf{x}^{\mathbf{T}}$ is a matrix of second moments and is positive definite. Hence its inverse exists. As a result m^* as proposed exists and is determined uniquely. □

Frictionless portfolio formation thus implies that assuming value additivity is equivalent to assuming that within the set of attainable payoffs one, and only one, stochastic discount factor exists. In other words, not only does a stochastic discount factor exist, it also can be constructed as a portfolio from the set of available payoffs and there is only one such portfolio in the payoff space. Complete markets are not necessary for this result.

(b) Arbitrage

The above result can be amended slightly if instead of assuming value additivity we assume *absence of arbitrage opportunties* (AOAO) While the no-arbitrage assumption in a sense is similar to the "law of one price" assumption, it is a little stronger. The law of one price only assumes that two basically equivalent assets should have identical prices. Absence of arbitrage implies that, if one asset first-order stochastically dominates another than it should have a higher price. Or, put differently, if one asset in no state has a lower payoff than a particular other asset but in some states has a higher payoff, then it should have a higher price. Stated formally:

Assumption 3 (Absence of Arbitrage Opportunities). For every payoff x in X, if *x(s) $ 0* for all s and $x(s) > 0$ for some s, then $p(x) > 0$.

A variant of the previous result then becomes:

RESULT 2 (ABSENCE OF ARBITRAGE AND THE STOCHASTIC DISCOUNT FACTOR). *Provided that Assumption 1 holds:*

> *Assumption 3* $\Leftrightarrow$ For all $x_i \in \mathrm{X}$: $E(m^* x_i) = p_i$, with $m^* \in \mathrm{X}$ and unique within X, and $m^* > 0$.

The slightly stronger assumption of no arbitrage thus implies the more specific result that the stochastic discount factor must also be positive in all possible states. A complete proof of this result can be found in Cochrane (1999). Here we prove the reverse implication in general but the implication only for complete markets.

Proof. First, $E(m^* x_i) = p_i$, and $m^* > 0 \rightarrow$ Absence of Arbitrage Opportunities. Clearly, if m^* is strictly positive in all contingencies and x_i is never negative and strictly positive in some contingencies then $p_i = E(m^* x_i) > 0$. Second, Absence of Arbitrage Opportunities
$\rightarrow E(m^* x_i) = p_i$, and unique $m^* > 0$. Although true for incomplete markets as well we prove this here only for complete markets. Note that equation (1) proves this somewhat informally already. Alternatively, consider that Assumption 3 implies Assumption 2. Hence we know that a unique m^* exists that may be negative. But say that for some state *s*,
$m^*(s) < 0$. Then consider the Arrow-Debreu security for state *s*: its $p_i = E(m^* x_i) < 0$, which contradicts the assumption of no arbitrage opportunities. □

It should be pointed out that the above result is quite general. It is possible of course to find a stochastic discount factor that is positive and uniquely determined in payoff space in the context of, say, the CAPM. But this would require an assumption of normality or quadratic preferences, together with other specific assumptions. Here, on the other hand, a sufficient assumption (other than frictionless portfolio formation) is absence of arbitrage opportunities. Within the context of frictionless markets, absence of arbitrage is implied by the very weak preference restriction of non-satiation: utility is strictly increasing in wealth (or consumption).

(c) Multiple Stochastic Discount Factors.

In a complete markets economy, *m* is unique. The reason is that we know from Result 2 that only one *m* exists within the payoff space. But with complete markets, the payoff space includes all possibilities so no *m* can exist outside of the payoff space.

When markets are incomplete, it is easy to construct additional stochastic discount factors. Consider for instance $m = m^* + \varepsilon$. Then $E(m x_i) = E[(m^* + \varepsilon) x_i] = p_i + E(\varepsilon x_i)$. In an incomplete markets economy, it is always possible to select infinitely many random outcome variables with mean zero that are independent of all feasible payoffs in the incomplete markets payoff space, so that $E(\varepsilon x_i) = 0$. Many of these *m*'s could be positive.

Here and in some of the further sections it is useful to use the idea of a *projection*. A projection provides the least squares forecast of a particular variable. Thus, $proj(y|1,x) = \hat{a} + \hat{b}x$, where $\hat{a}$ and $\hat{b}$ are the least squares estimates of a regression of *y* on *x*.

Consider the projection of m on the space of payoffs X. Then we can write that $m = proj(m|X) + \varepsilon$, $E(\varepsilon x_i) = 0$ for all i. Thus,

$$(5) \qquad p_i = E(mx_i) = E\{[proj(m|X) + \varepsilon]x_i\} = E[proj(m|X)x_i] = E(m^* x_i).$$

The last equality follows since the projection of m on the payoff space will lie within the payoff space and must thus be the unique discount factor that lies in the payoff space under assumptions 1 and 2 or assumptions 1 and 3.

If the stochastic discount rate generated by some particular asset pricing model with incomplete markets were m, then it may not be found as a portfolio from the available asset opportunities. However, equation (5) shows that it is always possible to construct a *mimicking portfolio* m^* of m which has the exact same pricing implications as m.

Interestingly, if one knew the true model and exact theoretical variable to represent m from the theoretical model, but if this variable were measured with some error, then sufficient data mining might produce the mimicking portfolio that, as is the case with portfolio returns, would likely be measured with less error and so might perform better than the true model!

(d) Systematic and Idiosyncratic Risk

Given a stochastic discount factor and using projections, it is possible to provide a natural definition of systematic and idiosyncratic risk. Set $x_i = x_i^* + \varepsilon_i$, where $x_i^* \equiv proj(x_i|m)$. Then:

$$p(\varepsilon_i) = E(m\varepsilon_i) = 0 \;, \quad p(x_i) = E(mx_i) = E(mx_i^*) = p(x_i^*).$$

Thus, the systematic risk is the risk that is perfectly correlated with the stochastic discount factor. It is found by projecting the return of asset i on m (without a constant so that ε_i need not have zero mean). The idiosyncratic risk is uncorrelated with m and is not priced as was shown.

3. STOCHASTIC DISCOUNT FACTORS AND BETA PRICING MODELS

We will explore here the connection between a stochastic discount factor representation, a beta-pricing representation, and a mean-variance efficiency representation and find that all three representations are equivalent.

(a) Stochastic Discount Factors and Beta-Pricing Models

We know from equation (5) that one can always find a mimicking portfolio for the "true" underlying stochastic discount factor that prices assets equally well.

$$(6) \qquad p_i = E(mx_i) \rightarrow p_i = E(m^* x_i) \Leftrightarrow 1 = E(m^* R_i),$$

where $m^* = proj(m|X)$. From equation (6) we can derive two useful results. First, if a risk free asset exists, its return would equal:

$$(7) \qquad 1 = E(m^* R_f) = E(m^*)R_f \rightarrow R_f = 1/E(m^*).$$

Second, we know that by construction m^* is the payoff on a portfolio of generally available assets. Now consider the return on the payoff m^*: $R_{m^*} \equiv m^*/p_{m^*}$. From equation (6) for asset m^* we then have:

$$(8) \qquad p_{m^*} = E(m^*)^2 \rightarrow R_{m^*} = m^*/E(m^*)^2 .$$

Returning to equation (6), use the definition of covariance to write:

$$(9) \qquad E(R_i) = \frac{[1 - Cov(m^*, R_i)]}{E(m^*)}$$

$$= \frac{1}{E(m^*)} + \left(\frac{Cov(m^*, R_i)}{Var(m^*)}\right)\left(-\frac{Var(m^*)}{E(m^*)}\right) .$$

Clearly, we already have a beta pricing formulation at this point. However, to obtain a more standard formulation, use equation (8) to relate the return on any asset to the return on the stochastic discount factor mimicking portfolio. Apply equation (9) to the stochastic discount factor mimicking portfolio and use equation (8) to obtain:

$$(10) \qquad E(R_{m^*}) = \frac{1}{E(m^*)} - \frac{Var(R_{m^*})}{E(R_{m^*})} .$$

Rewrite equation (9) to transform to returns on the mimicking portfolio and then combine equations (9) and (10) to produce:

$$(11) \qquad E(R_i) = \frac{1}{E(m^*)} - \left(\frac{Cov(R_{m^*}, R_i)}{Var(R_{m^*})}\right)\left(\frac{1}{E(m^*)} - E(R_{m^*})\right).$$

If a risk free asset exists then from equation (7) we can write:

$$(12) \qquad E(R_i) = R_f + \beta_{im^*}[E(R_{m^*} - R_f)] \quad , \quad \text{where } \beta_{im^*} \equiv \frac{Cov(R_{m^*}, R_i)}{Var(R_{m^*})} .$$

Or, converting gross returns into returns:

$$(13) \qquad \mu_i = r_f + \beta_{im^*}(\mu_{m^*} - r_f) .$$

Note that the steps that lead to equation (13) can also be reversed. So the existence of a single beta pricing formulation, with beta related to the covariance between asset return and the mimicking portfolio return, implies the existence of an m^* that prices all assets.

In summary:

RESULT 3 (STOCHASTIC DISCOUNT FACTORS AND BETA REPRESENTATIONS). *Iff. a stochastic discount factor formulation applies, a one-beta model exists with as its factor the return on the stochastic discount factor mimicking portfolio:*
$p_i = E(mx_i) \Leftrightarrow \mu_i = r_f + \beta_{im^*}(r_f - \mu_{m^*})$ *, with* $\beta_{im^*} \equiv Cov(r_{m^*}, r_i)/Var(r_{m^*})$.

Thus, assumptions 1 and 2 (or 3) are seen to produce a CAPM-type model in which the covariance with one particular portfolio return is sufficient to price each asset. The key portfolio here is the portfolio that mimics the stochastic discount factor *m*. This portfolio, of course, in general is more difficult to obtain than the market portfolio used in the CAPM.. Additionally, the portfolio has a negative risk premium as follows from equation (10). This makes sense since high returns when the future is discounted less are more valuable.

(b) Beta Pricing Models and Stochastic Discount Factors

What if there are more betas or when the beta is not the beta with the mimicking portfolio? We will see next that a linear factor model (such as the APT) is equivalent to a stochastic discount factor that is linear in the factors.

Define **f** as a demeaned column vector of the *k* factors and **b** as a column vector of *k* constants. (Note that demeaning the factors is without loss of generality as the factor means appropriately weighted can be added to the constant). Additionally, define $\boldsymbol{\beta_i}$ as the column vector of sensitivities of asset *i* to the *k* factors and $\boldsymbol{\lambda}$ as the column vector of risk premia on the *k* factors. Then:

RESULT 4 (LINEAR STOCHASTIC DISCOUNT FACTORS AND GENERAL BETA REPRESENTATIONS). *The following two representations are equivalent:*

$$(A) \qquad \mathrm{E}(mR_i) = 1 \,, \; m = a + \mathbf{b}^{\mathbf{T}}\mathbf{f}$$

$$(B) \qquad E(R_i) = \alpha + \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\beta}_{\mathbf{i}} \,,$$

$$where \;\; \alpha = 1/a \;, \quad \text{and} \quad \lambda = -\alpha \mathrm{E}(\mathrm{m}\mathbf{f}) \;.$$

Proof. From (A), and similar to equation (9), we get that:

$$(14) \qquad E(R_i) = \frac{[1 - Cov(m, R_i)]}{E(m)} = \frac{1}{a} - \frac{E(R_i \mathbf{f}^{\mathbf{T}})\mathbf{b}}{a} \;.$$

The second equality follows from using (A) and since the means of $\mathbf{f}$ are equal to zero. To put the $\beta_\mathbf{i}$ into equation (14), recall that

$$(15) \qquad \beta_i \equiv E(\mathbf{f}\,\mathbf{f}^{\mathbf{T}})^{-1} E(\mathbf{f}\, R_i) \;.$$

Substitute equation (15) into equation (14) to obtain

$$(16) \qquad E(R_i) = \frac{1}{a} - \beta_\mathbf{i}^\mathbf{T} \frac{E(\mathbf{f}\,\mathbf{f}^{\mathbf{T}})\mathbf{b}}{a} \;.$$

It is now easy to check with the appropriate substitutions that we obtain representation (B). Vice versa, it is straightforward to go from equation (16) back to equation (14) and then to representation (A). □

Note that the representations are not unique. For instance, we have seen that given incomplete markets, you can add a random variable orthogonal to returns to m, leaving pricing implications unchanged. And, in representation (B), adding risk factors with zero β or with zero λ would leave pricing implications unchanged as well. Note also that, from a practical perspective, Result 4 tells you how to discount cash flows if the APT is assumed to hold. Further, note that from Result 4 we can write $\lambda_k / R_f = p(f_k)$. I.e., the present value of the risk premium on factor k is equal to the price of the factor. Lastly, note that we know from Result 3 for instance that we could combine the different factors into just one factor, m.

(c) Stochastic Discount Factors and Mean-Variance Efficiency

We will assume here for simplicity that a risk free asset exists. The results in this section hold also for the more general case as shown in Cochrane (1999). We start again with the assumption that a stochastic discount factor representation holds for all assets. We can then write, using equations (6) and (7), and using the linearity property of the covariance operator:

$$(17) \qquad E(mR_i) = 1 \;\Leftrightarrow\; E(R_i) = R_f - \frac{Cov[(m^*)/p_{m^*}), R_i]}{E[(m^*)/p_{m^*}]} \,,$$

if a risk free asset exists. Using the definition of the correlation coefficient of the returns on asset *i* and the mimicking portfolio:

$$(18) \qquad E(mR_i) = 1 \;\Leftrightarrow\; E(R_i) = R_f - \rho_{im}\,\sigma_{R_i}\,\frac{\sigma(R_{m^*})}{E(R_{m^*})} \,.$$

By employing the fact that $-1 \le \rho_{im} \le 1$ it is possible to construct a mean-variance frontier:

$$(19) \qquad E(mR_i) = 1 \;\Leftrightarrow\; |E(R_i) - R_f| \le \sigma_{R_i}\left(\frac{\sigma(R_{m^*})}{E(R_{m^*})}\right) .$$

Figure (1) shows the wedge-shaped mean-variance frontier. Clearly the risk free asset is on the frontier. This is also true for the mimicking portfolio which follows from the fact that $\rho_{mm} = 1$. Equation (18) shows that indeed the mimicking portfolio is on the mean-variance frontier. It is however on the inefficient part of the frontier. This provides the explanation for why the risk premium in equation (17), for instance, is negative. Note that the term in large parentheses equals the "maximal Sharpe Ratio."

Combinations of the risk free asset and the mimicking portfolio trace out the mean-variance frontier. This is clear from the fact that linear combinations of these two assets are either perfectly positively correlated or perfectly negatively correlated. Thus for all portfolios *i* that are linear combinations of the risk free asset and the mimicking portfolio, $|\rho_{im}| = 1$. As a result, it is possible to write either m^* or $R_{m^*} = m^*/E[(m^*)^2]$ as linearly related to any mean-variance efficient return, R_{MVF}. So any frontier return is sufficient for pricing. Thus,

RESULT 5 (STOCHASTIC DISCOUNT FACTORS AND THE MEAN VARIANCE FRONTIER). *The existence of a stochastic discount factor representation is equivalent to (1) $R_{MVF} = a + bm^*$ and to the beta formulation of beta with any mean-variant efficient return (2) $E(R_i) = R_f + \beta_{i\,MVF}\,[E(R_{MVF}) - R_f]$.*

Our proof here assumed existence of a risk free asset, but, as stated previously, the result holds also if no such asset exists. Of course we already knew the result. Roll (1976) proved that a single beta representation exists if and only if the benchmark return is on the mean-variance frontier. Thus, Result 5 follows from Roll's result plus Result 3 (which relates the stochastic discount factor representation to a single beta pricing model.

Note that the expression in equation (19) is a key element in the "Hansen-Jagannathan Bound" which relates excess returns to volatility in fundamentals. Also note that in Figure 1 it is easy to indicate the idiosyncratic risk of any asset. As discussed in section 2, the idiosyncratic risk is the part not correlated with m or m^*. Hence it is causing the

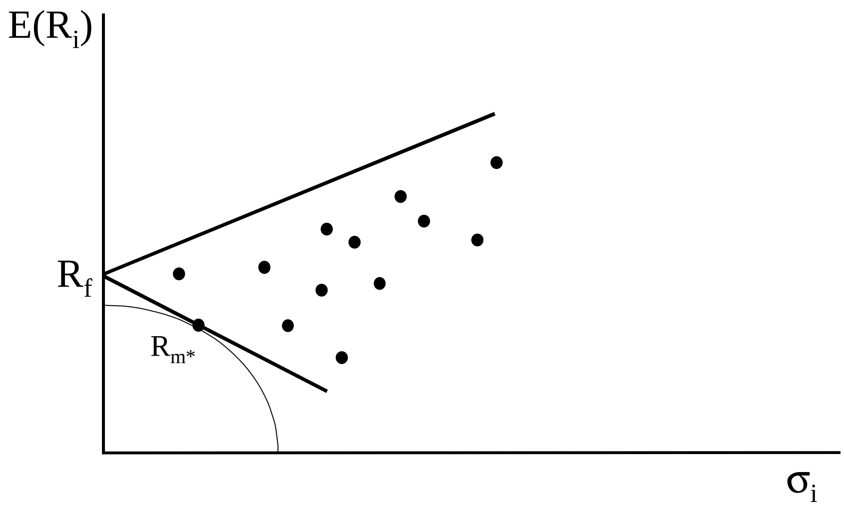

Figure 1
Stochastic Discount Factor and Portfolio Frontier
The return on any mean-variance efficient portfolio
must be perfectly correlated with m*.

variance in excess of the variance of the efficient asset with equal mean:

(20) $$x_i = x_i^* + \varepsilon_i, \text{ where } x_i^* \equiv proj(x_i|m) \rightarrow \sigma_i^2 = \sigma_{i^*}^2 + \sigma_\varepsilon^2 .$$

Lastly, Figure 1 shows that R_{m^*} can be found as the minimum feasible second moment asset as shown in Cochrane (2001).