Chapter VI. Valuation in Complete Market Economies

The objective in this chapter is to present results that are derivable without using specific assumptions about the structure of the economy. We discuss results that follow from general equilibrium arguments.

1. COMPLETE MARKETS

Up to this point we have often used the concept of *perfect* markets. Perfect markets in brief are perfectly competitive and frictionless. This concept may apply in a certainty as well as an uncertainty environment. The idea of *complete* markets is completely different. It applies only to an uncertainty environment and does not presume perfect competition or absence of frictions.

(a) Definition

A complete set of markets refers to the situation where every contingency is insurable. That is, one could insure against every possible state of the economy at a market price. More specifically, for every possible outcome of the economic process, it is possible in advance to construct a portfolio in such a way that you receive a real return under that outcome, independent of what you get under the other possible outcomes. So, in theory, you can make a list of all the possibilities and calculate in advance how much you will be able to consume for each possibility if it materializes. Of course, you could then eliminate all uncertainty by guaranteeing identical consumption levels for each possible state. You likely would not do so, though, since insurance for different states will have different costs. Old-style economists might say that complete markets change an “uncertain” environment to a merely “risky” environment, but we will continue to use the terms risk and uncertainty interchangeably to describe environments with complete or incomplete markets.

(b) Arrow-Debreu Securities

Arrow-Debreu securities are defined as paying one unit of real wealth in one particular state of the economy and zero in every other possible state. Clearly, if Arrow-Debreu securities are available for each state, markets are complete. Other terms for Arrow-Debreu securities are *state contingent claims* and *pure, state* or *primitive securities*.

These securities don’t exist of course in practice. But the question is: could they be created in principle? The answer depends on the set of existing regular (“primary”) securities. For instance, consider an economy with two possible states – good and bad. If we have only one security, say a riskless one, we would get the same outcome in both states and would not be able to insure one state independent of the other. If we have two securities – one riskless, one risky – then independent insurance is possible for each state. You could get a return only in the good state by buying the risky asset and short-selling the riskless asset in such quantities that the payoff in the bad state is zero.

In general, for markets to be complete there need to be enough securities with linearly independent payoffs to insure all different possible states. Thus:

RESULT 1 (MARKET COMPLETION). *In an economy with frictionless financial markets and* s *possible states, markets are* complete *if* s *securities exist whose payoffs are linearly independent.*

Proof. Consider s states. A particular security will have potentially different payoffs for each state. If we have s securities, we can construct an $s \times s$ matrix $\mathbf{X}$ which has as its x_{ij} element the payoff (per share) of security i in state j. We need to show that holding a judiciously selected portfolio of securities allows creation of an Arrow-Debreu security for any particular state. Define $s \times s$ matrix $\mathbf{P}$ such that element p_{ij} represents the number of shares of security j needed to create an Arrow-Debreu security for state i. Thus, column i gives the portfolio weights (possibly negative, and not necessarily summing to zero or one) needed to construct the Arrow-Debreu security for state i. Any such portfolio is feasible due to the assumed absence of frictions. In particular, short sales are possible without transactions costs.

If we can find $\mathbf{P}$ such that $\mathbf{X\,P = I}$, where $\mathbf{I}$ is the identity matrix then we are done since then portfolio formation based on the existing securities allows creation of an Arrow-Debreu security for every state. Specifically, the nth column of the identity matrix indicates a zero payoff for the nth created security in all states except for the nth state when the payoff is one. If $\mathbf{X\,P = I}$ then $\mathbf{P = X^{-1}}$. Thus if (and only if) the inverse of $\mathbf{X}$ exists it is possible to choose $\mathbf{P}$ to create all Arrow-Debreu securities. But, by assumption, the columns of $\mathbf{X}$ are linearly independent so that the inverse must exist. □

(c) Implications of Complete Markets

The assumption of complete markets allows extension of the standard Arrow-Debreu model for a static competitive economy under certainty, to multi-period cases with risk. The extension is trivial for complete markets since now any state contingent payoff, no matter how far into the future, has a specific price and can be treated just like any other commodity. Thus, the two fundamental welfare theorems hold in a complete markets environment with uncertainty and multiple periods. Without proof:

RESULT 2 (PARETO EFFICIENCY). *(a) Given the standard preference and technology assumptions for a perfectly competitive economy with complete markets, the market outcome is Pareto Efficient. (b) Given any Pareto Efficient outcome, prices exist for all commodities (including all Arrow-Debreu securities) such that the outcome can be sustained in a perfectly competitive economy with complete markets.*

Note that Result 2(b) applies in the case where a social planner maximizes any weighted average (with non-negative weights) of the expected utilities of all consumers subject to using all available resources.

To consider the implications of complete markets for *risk sharing,* we take the simple case of a two-period model with time-additive utility, although the results apply more generally. Investor i faces the following decision problem:

$$(1) \qquad \max_{\{c_i^0,\, c_i(s)\}_{s=1}^S} u_i^0(c_i^0) + \beta\, E[u_i(c_i)], \qquad E[u_i(c_i)] = \sum_{s=1}^{S} \pi_s\, u_i[c_i(s)]$$

$$(2) \qquad \text{Subject to}: \quad w_i^0 + \sum_{s=1}^{S} p_s y_i(s) = c_i^0 + \sum_{s=1}^{S} p_s c_i(s) .$$

Here π_s represents the probability that state *s* occurs; w_i^0 indicates initial wealth and $y_i(s)$ indicates the income to be received by investor *i* contingent on state *s* occurring. The price p_s denotes the price of the Arrow-Debreu security for state *s* , or *state price*, paying one consumption good in state *s* and nothing in other states.

Substitute equation (2) into equation (1) to eliminate the non-stochastic consumption level for the first period, c_i^0. The first-order conditions for consumption choice in each state *s* become:

$$(3) \qquad p_s\, u_i^{0\,\prime}(c_i^0) = \beta\, \pi_s\, u_i^{\prime}[c_i(s)], \qquad \text{for all } s .$$

The marginal cost of ensuring additional consumption in state *s* is the state price valued at the marginal utility of consumption in the period when you pay the state price. The marginal benefit is the discounted marginal utility that will prevail in state *s* times the probability that this state will occur.

Rewriting the conditions in equation (3) by considering state *s* and some other state *v* produces:

$$(4) \qquad \frac{\pi_s\, u_i^{\prime}[c_i(s)]}{p_s} = \frac{\pi_v\, u_i^{\prime}[c_i(v)]}{p_v} ,$$

which of course holds for any *s* or *v*. Thus, the individual equalizes the “bang for the buck” in every state: probability times marginal utility of consumption for the state per “dollar” to insure the state are equal in every state.

The optimal risk sharing arising from complete markets can now be shown by considering equation (4) for different investors *i* and *j* :

$$(5) \qquad \frac{p_v / \pi_v}{p_s / \pi_s} = \frac{u_i^{\prime}[c_i(v)]}{u_i^{\prime}[c_i(s)]} = \frac{u_j^{\prime}[c_j(v)]}{u_j^{\prime}[c_j(s)]} ,$$

for all *i* and *j* and for all *v* and *s* . Thus, if expectations are homogeneous in the sense that all consumers perceive equal probabilities of states occurring so that $\pi_{is} = \pi_s$ for all *i*, then the marginal rates of substitution (relative marginal utility of consumption) between any two states are equalized for all investors. The upshot is that there are no *ex post* regrets: it is not expected marginal rates of substitution that are equalized, it is actual marginal rates of substitution that are equalized! No profitable trades remain between any two individuals, which is of course a direct implication of Pareto Efficiency in this context. Summarizing the result:

RESULT 3 (RISK SHARING). *Given a perfectly competitive economy with complete markets and homogeneous expectations, the realized marginal rates of substitution for consumption between all states (and periods) are equal for all rational individuals.*

Note that combining equations (3) and (5) implies that:

$$(6) \qquad p_s / \pi_s = \frac{\beta_i u_i' [c_i(s)]}{u_i^{0\,\prime}(c_i^0)} = \frac{\beta_j u_j' [c_j(s)]}{u_j^{0\,\prime}(c_j^0)} ,$$

which states the result in terms of realized marginal rates of substitution in consumption between periods. If we were to assume Constant Relative Risk Aversion (CRRA) utility for all investors we would get from equation (6) that: $\beta_i [c_i(s) / c_i^0]^{-\gamma_i} = \beta_j [c_j(s) / c_j^0]^{-\gamma_j}$, where γ represents the coefficient of relative risk aversion. If both impatience and risk aversion of the individuals is equal then risk sharing implies that the growth rates of consumption are equalized, no matter what the state of the economy. This sort of implications is often discussed as a benchmark for deciding to what extent risk sharing between countries or regions is complete.

The risk sharing based on market prices for the state contingent claims hints at the Fisher Separation result that we considered in Chapter I for intertemporal preferences under certainty. As we will see here, a Fisher Separation result also applies under uncertainty. Figure 1 shows that, if we put consumption at different states on the axes (rather than consumption at different times as we did in Chapter I), then the market-determined price ratio (per unit probability) is again equal for all individuals given homogeneous expectations. Thus, each individual will choose the same marginal rate of substitution in consumption as we saw algebraically in equation (5). This implies that, given complete markets, individual preferences related to risk are in a certain sense equalized in equilibrium. Thus, risk preferences of management or shareholders should be irrelevant for decision making in a complete markets environment (with perfect competition), decisions should be made on an expected net present value basis.

To see this result more clearly, consider the *indirect utility function* of investor *i* who may be involved in a corporate decision:

$$(7) \qquad v_i [\{p_s\}_{s=1}^S , w_i^0 + \sum_{s=1}^S p_s y_i(s)] .$$

The arguments in the indirect utility function are of course (only) the set of market prices (here only including the state prices) and life time wealth. Obviously, maximum utility, v_i, is increasing in life time wealth. Now consider a choice between different investment projects yielding payoffs $y_i(s)$ in the different states. Clearly,

$$(8) \qquad \sum_{s=1}^S p_s y_i^*(s) > \sum_{s=1}^S p_s y_i(s) \rightarrow$$
$$v_i [\{p_s\}_{s=1}^S , w_i^0 + \sum_{s=1}^S p_s y_i^*(s)] > v_i [\{p_s\}_{s=1}^S , w_i^0 + \sum_{s=1}^S p_s y_i(s)].$$

Thus, the feasible project with the highest expected net present value, $\Sigma\, p_s y_i(s)$, should be preferred, irrespective of the risk preferences of the constituents. In summary:

> **RESULT 4 (FISHER SEPARATION).** *Given a perfectly competitive economy with complete markets and homogeneous expectations, real investment decisions are based on expected net present value and are independent of the risk preferences of management or shareholders.*

The assumption of perfect competition is necessary since the price of any commodity produced should be taken as given in the indirect utility function. If a firm has price-setting power, its investment decisions may affect the price of a particular commodity. If so, different individuals will have different preferences for the investment project. Similarly, complete markets are necessary for the Fisher Separation result. If a state price does not exist for a particular state, then the payoff of the project in that state will be evaluated differently by different investors as is shown in Figure 2.

The proper valuation for an asset or project, $\Sigma\, p_s y_i(s)$, may be given a more familiar interpretation by first specifying the expected return on a state contingent claim. Taking return conventionally as payoff divided by initial investment, we find the expected return on state contingent claim *s* as:

$$(9) \qquad 1 + r^e(s) = \frac{\pi_s 1 + (1-\pi_s) 0}{p_s} = \frac{\pi_s}{p_s} .$$

Thus, using equation (9), we can express the value of a project in a complete market with net payoffs $y_i(s)$ in each state in expected net present value form:

$$(10) \qquad \sum_{s=1}^{S} p_s y_i(s) = E\left(\frac{y_i(s)}{1 + r^e(s)}\right) .$$

Note that the expectation is taken over both net payoffs and state contingent claim returns.

It is straightforward to add a dynamic element to the complete markets model. For instance, we can separate the states into states prevailing in different years. So:

$$(11) \qquad s = 1, \ldots\ldots, S = 1, \ldots, s_1, s_1 + 1, \ldots, s_2, s_2 + 1, \ldots, \ldots.., s_{T-1} + 1, \ldots S ,$$

where subscripts indicate years. Then, for states between s_{t-1} and s_t:

$$(12) \qquad [1 + r^e(s)]^t = \pi_s / p_s , \qquad \text{for} \quad s_{t-1} < s \le s_t .$$

So the multi-period equivalent of equation (10) becomes:

(13) $$\sum_{s=1}^{S} p_s y_i(s) = \sum_{t=1}^{T} E\left(\frac{y_i(s_{t-1} < s < s_t)}{[1 + r^e(s_{t-1} < s < s_t)]^t}\right) ,$$

where the expectation is taken over the relevant *s*.

It must be pointed out that the number of states needed to describe a multi-period situation typically must grow exponentially. The reason is that, in many set-ups, the history of how a particular time *t* is reached, matters. For instance, if output was low in period 1, low output in period 2 has different implications than if output was high in period 1. Thus, if *n* events can occur during a period, then the number of states necessary to describe the possible states for period 2 would be *nxn*. Since the total number of states *S* can be any finite number, the exponential progression in number of states necessary to describe a multi-period problem is not a theoretical problem.

(d) Some Further Properties of State Prices

Consider purchasing a state contingent claim for every state. This yields exactly one unit of real wealth one period from now, no matter what happens. Thus, this investment strategy should yield the risk free real interest rate. Or:

(14) $$\sum_{s=1}^{S} p_s = \frac{1}{1 + r} .$$

One may rewrite the expression for the state price obtainable from equation (9) as:

(15) $$p_s = \frac{\pi_s}{1 + r^e(s)} = \pi_s \left(\frac{1}{1 + r}\right)\left(1 - \frac{r^e(s) - r}{1 + r^e(s)}\right) .$$

Equation (15) displays state prices as being determined by three different components. First, the probability that the state occurs; second, the risk-neutral market discount factor; and third, a risk adjustment factor.

It should be clear that any contingent security, such as an option, can be priced with Arrow-Debreu securities by employing the expression $\Sigma\, p_s y_i(s)$. The trick is to specify the states in which the option is “in the money” and the payoffs for these states.

(e) Applications and Exercises

1. Consider an economy which faces two possible states, state *b* with probability π and state *g* with probability *1*-π . Two securities exist with prices $p_1 = p_2 = 1$. Security *1* pays *1* unit in state *b* and pays *2* units in state *g*. Security *2* pays *0* units in state *b* and pays *4* units in state *g*.

(a) Explain whether or not markets are complete in this economy.

(b) Derive the state prices for this economy. Calculate the risk free rate in this economy.

(c) Suppose that Security *3* is issued which pays *2* units in state *b* and *3* units in state *g*. What would be its price?

(d) Explain what would happen to the state prices if the supply of Security 1 decreased.

(e) What condition on the probability π of state *b* occurring would imply that the economy is risk averse?

2. For the following statements, explain whether the statement is true, false, or uncertain.

(a) A riskless asset need not exist in a complete markets economy.

(b) Firms in a complete markets economy must be perfectly competitive.

(c) Arrow-Debreu securities must be worth less than 1 unit of real wealth.

(d) Since marginal rates of intertemporal substitution are equalized for all states, investors face no risk in a complete markets economy.

3. Are markets necessarily complete in the CAPM? If not what would be the possible implications of completing markets in a CAPM context?

2. EFFECTIVE MARKET COMPLETION

There are the number of instances in which markets are not formally complete but in which outcomes are *as if* markets are complete. In this case we say that markets are *effectively complete*.

(a) Options

When options can be written on primary securities (not to be confused with primitive securities), an additional state can be covered for each strike price. In fact, when the state is described by a particular numerical outcome and a strike price exists for each possible outcome, then the options written on the one primary security together with the security itself are sufficient to complete the market. See Ross (1976b). Thus, even with a large number of different states, markets can be completed with derivative securities based on a relatively small number of primary securities. Note that here we are still dealing with complete markets rather than effectively complete markets.

(b) The Aggregate Endowment as a Sufficient Statistic

Rearranging equation (5) yields:

$$(16)\qquad \frac{u_i^{/}[c_i(v)]}{u_j^{/}[c_j(v)]} = \frac{u_i^{/}[c_i(s)]}{u_j^{/}[c_j(s)]}\ , \qquad \text{for all } i, j \text{ and } s, v\ .$$

The ratio of marginal utilities of consumption across any two investors is equal for all states. Or, in other words, the relative marginal utilities of consumption across individuals are independent of the state and are, thus, non-stochastic.

Define the aggregate endowment in state *s* as $C(s) = \Sigma\, c_i(s)$. If, say, $C(v) > C(s)$ then there must be at least one investor *i* for whom $c_i(v) > c_i(s)$. For this investor $u_i^{/}[c_i(v)] < u_i^{/}[c_i(s)]$. But then, equation (16) implies that, for any other investor *j*, $u_j^{/}[c_j(v)] < u_j^{/}[c_j(s)]$ so that for any other investor $c_j(v) > c_j(s)$. Similar reasoning applies if $C(v) = C(s)$ or if $C(v) < C(s)$. Thus,

$$(17)\qquad C(v) \ \begin{matrix} > \\ = \\ < \end{matrix}\ C(s) \quad \rightarrow \quad c_i(v) \ \begin{matrix} > \\ = \\ < \end{matrix}\ c_i(s)\ , \qquad \text{for all } i \text{ and } v, s\ .$$

It follows, for instance, that two different states yielding identical aggregate endowments are for all practical purposes identical since they lead to equivalent (consumption) actions. States may be ranked based on the aggregate endowment that they produce. Any investor will consume more as the aggregate endowment is larger. Hence there is a one-to-one mapping between the aggregate endowment and individual consumption which implies formally that:

$$(18)\qquad c_i = f_i(C)\ , \quad f_i^{/}(C) > 0\ , \qquad \text{for all } i\ .$$

Note that the result follows only when preferences are not state dependent so that in equation (16) the marginal utilities depend on consumption only.

A key implication of all this is that the aggregate endowment is a sufficient statistic for all actions and that states are appropriately summarized by their resulting aggregate endowment only. To summarize:

> **RESULT 5 (SUFFICIENCY OF AGGREGATE ENDOWMENT).** *In the two-period economy of equation (1) and (2), the aggregate endowment outcome* C(s) *is a sufficient statistic for the state* s*.*

As a result, markets may be effectively completed by insuring aggregate endowment levels only.

(c) Dynamic Trading

Without proof we summarize a result for completing markets in a dynamic (multi-period) environment. For a formal analysis see Kreps (1982) or Huang and Litzenberger (1988, Chapter 7). As stated previously, the number of states to consider in a multi-period model increases exponentially with the number of periods. If markets are completed conventially, Pareto Efficiency is accomplished in the competitive multi-period economy. One feature of the competitive equilibrium in this case is that trading beyond time *0* does not occur if expectations are homogeneous. The intuitive reason is that relative marginal utilities across investors are the same ratio in each state; including states that will occur far into the future. As the probability that a particular state will occur is revised when information comes in over time, this revision is identical for all investors given homogeneous expectations and so there is no incentive to trade at dates beyond time *0*.

An alternative scenario is one in which markets are effectively completed with a much smaller number of securities but in which securities are traded actively beyond time *0*. Consider a multi-period economy where individuals have time-additive expected utility preferences and have homogeneous expectations. The following result applies to this economy: If enough "long-lived" (that is, existing in all periods) securities exist to complete markets in every two-period segment of this economy then markets can be "dynamically" completed by trading of the long-lived securities in each period.

The intuition is best understood by visualizing a decision tree where the number of nodes at the end of each period equals the number of states (to be covered to complete the market in a two-period segment). An investor just needs to anticipate for each state at the end of the period how current and future opportunities are affected, and to hedge herself accordingly. This is possible if markets are complete with respect to the states at the end of the upcoming period only. Once the state is realized at the end of the period, the investor may reallocate her portfolio to reposition herself for the next period in light of any new information that was inherent in the realization of the state.

(d) Preference Restrictions

Cass and Stiglitz (1970) have shown that any investor with HARA (Hyperbolic Absolute Risk Aversion) utility hold the same two assets, the riskless asset and a risky portfolio, no matter what their level of wealth. HARA utility is given as $u(c) = [b/(b-1)](a+bc)^{1-(1/b)}$, with the properties that for $\lim b \to 1$ we obtain the logarithmic form (with additive constant a) and for $\lim b \to 0$ we obtain the negative exponential form. Furthermore, the function includes the power utility function (for a = 0) and the quadratic utility function (for b = -1) as special cases.

For a large class of reasonable utility functions we therefore have a two-fund separation result. It implies that individual investors need only pick one mutual fund for their risky asset investments which will not change as their wealth changes. This result has no direct aggregate market implications since each HARA investor may have different utility parameters and would hold a different risky portfolio. Simplifying market implications arise if the b parameters are set equal across all investors.

We here prove the two-fund separation result and show that markets are always effectively complete if all individuals (indexed by k) have the following preferences:

$$u_k(c_k) = [b/(b-1)](a_k + bc_k)^{1-(1/b)}. \tag{20}$$

The context is a standard competitive static model with incomplete markets. Each individual maximizes $E[u_k(c_k)]$, subject to:

$$c_k = \bar{w}_k[1 + r_f + \sum_{i=1}^{n} s_{ik}(r_i - r_f)]. \tag{21}$$

The first-order conditions for this decision problem are:

$$(22) \qquad E[u_k^{/}(c_k)\,(r_i - r_f)] = 0 \ , \quad \text{for all } i \text{ and } k.$$

Substituting the specific utility function in equation (20) and the budget constraint in equation (21) into equation (22) produces:

$$(23) \qquad E\left(\{ a_k + b\overline{w}_k [1 + r_f + \sum_{i=1}^{n} s_{ik}\,(r_i - r_f)] \}^{-1/b} (r_i - r_f) \right) = 0 \ , \quad \text{for all } i \text{ and } k$$

Now take any other investor *j* and compare the portfolio decisions of investors *j* and *k*. It is easy to verify that:

$$(24) \qquad \frac{\overline{w}_k s_{ik}}{a_k + b\overline{w}_k(1 + r_f)} = \frac{\overline{w}_j s_{ij}}{a_j + b\overline{w}_j(1 + r_f)} \ , \quad \text{for all } i \ .$$

To verify, substitute equation (24) into equation (23) for all *i*. Then the non-random terms $a_k + b\overline{w}_k(1 + r_f)$ and $a_j + b\overline{w}_j(1 + r_f)$ can be factored out of the term in braces in equation (23) and can thus be dropped since the right-hand side of equation (23) is zero. The resulting expression is exactly the first-order condition for asset *i* but as is relevant for investor *j*. Thus equation (24) is correct.

Aggregate equation (24) over all risky assets *i*. This yields:

$$(25) \qquad \frac{\overline{w}_k (1 - s_{0k})}{a_k + b\overline{w}_k(1 + r_f)} = \frac{\overline{w}_j (1 - s_{0j})}{a_j + b\overline{w}_j(1 + r_f)} \ ,$$

since the sum of all risky investment shares for any individual is equal to the investment share in the riskless asset. Combining equations (24) and (25) gives:

$$(26) \qquad \frac{s_{ik}}{1 - s_{0k}} = \frac{s_{ij}}{1 - s_{0j}} \ , \quad \text{for all } i.$$

It is now easy to see, for instance by dividing both sides of equation (26) by the equivalent version for *i'*, that both investors hold the risky assets in the same ratios. Thus, all risky assets are held in the same proportions by all investors. Thus, any investor cares just about two assets, a risky mutual fund and the risk free asset, which proves the two-fund separation result.

Since all investors hold the same risky "mutual fund", and no other risky assets are held by anyone, the mutual fund must be the market portfolio. Suppose now that a complete set of state contingent claims are created which are in zero net supply. These claims do not change the market portfolio since they are in zero net supply. Hence, investors are indifferent about the introduction of these state securities. The allocation is not affected in any way. In other words, markets are always effectively complete when preferences of all investors are as indicated in equation (20). Note that a similar argument can be made for general utility functions, but when *every investor has the same utility function*: each investor holds the same portfolio which is thus the market portfolio (of all assets including the riskless asset). Introduction of assets in zero net supply to complete the market is accordingly immaterial and markets are always effectively complete.

(e) Applications and Exercises

1. Construct a simple example to show that options on primary assets with various strike prices help to complete markets.

2. Construct a simple example to show that asset trading in a 3-period economy effectively completes markets when (exactly) enough assets with independent payoffs exist to cover all states between any two periods.

3. Prove that the price of any financial asset in a complete markets economy may be written as:

$$p = \sum_{k=1}^{K} p_k \, E(x|c=k) \ ,$$

   where p_k is the price of an asset paying one unit of real wealth only when aggregate consumption is equal to k ; x indicates the payoff of the asset; the last part of the equation then represents the payoff of the asset *conditional* on aggregate consumption being equal to k .

4. Since 2-fund separation holds for HARA preferences, why doesn't a CAPM-type model hold when all investors have HARA preferences? Explain.

3. REPRESENTATIVE INVESTORS

(a) Existence of a Representative Investor when Markets are Complete

We return to the two-period model with, for all investors, homogeneous beliefs and time-additive utility that is not state dependent. The model is described for investor *i* in equations (1) and (2). The first-order conditions are given in equation (3). A complete markets assumption guarantees that the outcome is Pareto Efficient. The "second welfare theorem", Result 2(b) above, implies then that any market outcome can be generated as the result of a "social planner" optimization problem. The social planner maximizes a *weighted average* of the utilities of all individuals in the economy.

Define the following functions:

(27) $$U^0(C^0) = \max_{\{c_i^0\}_{i=1}^K} \left(\sum_{i=1}^K \lambda_i u_i^0(c_i^0) \right) ,$$

(28) $$U[C(s)] = \max_{\{c_i(s)\}_{i=1}^K} \left(\sum_{i=1}^K \lambda_i u_i[c_i(s)] \right) , \quad \text{for all } s,$$

subject to the aggregate resource restrictions:

(29) $$\sum_{i=1}^K c_i^0 = C^0 \ ; \quad \sum_{i=1}^K c_i(s) = C(s) \ , \quad \text{for all} \quad s \ .$$

The weights on the individuals are chosen to be equal to the inverse of their marginal indirect utility of initial wealth (which equals the inverse of the Lagrangian multiplier on their budget constraint). The envelope theorem implies that:

(30) $$\lambda_i^{-1} \equiv v_i^{\prime}(\bar{w}_i) = u_i^{\prime}(c_i^{0\,*}) .$$

Since the Pareto Efficient outcome obtained in this way is the competitive market outcome for the initial allocation $\{\bar{w}_i\}_{i=1}^K$, it follows that:

(31) $$U^{0\prime}(C^0) = \sum_{i=1}^K \lambda_i u_i^0(c_i^{0\,*}) \frac{dc_i^0}{dC^0} = \sum_{i=1}^K \frac{dc_i^0}{dC^0} = 1,$$

as follows from equations (27), (30) and (29). Similarly,

(32) $$U^{\prime}[C(s)] = \sum_{i=1}^K \lambda_i u_i^{\prime}[c_i(s)^*] \frac{dc_i(s)}{dC(s)} = \frac{p_s}{\beta \pi_s} \sum_{i=1}^K \frac{dc_i(s)}{dC(s)} = \frac{p_s}{\beta \pi_s} ,$$

for all s, as follows from equation (3).

We may then consider an agent with utility functions given by equations (27) and (28) a *representative investor*. The representative investor is further assumed to be endowed with all initial wealth. It is easy to check that the first-order conditions analogous to equation (3) hold for the representative investor by using equations (31) and (32). Or, reversing causation since the aggregate consumption realizations for each state are predetermined, state prices are given as in the "decentralized" investor model of equations (1) and (2):

(33) $$p_s = \pi_s \beta U^{\prime}[C(s)] / U^{0\prime}(C^0) .$$

Thus, state prices may be inferred from the marginal rate of intertemporal substitution of the representative investor. An important proviso is that for the representative investor created here, the utility function will change with the distribution of initial endowments. (The λ_i vary with initial wealth). In summary:

> **RESULT 6 (REPRESENTATIVE INVESTOR WITH COMPLETE MARKETS).** *Given time-additive preferences that are not state dependent, homogeneous expectations, and complete markets, a representative investor always exists. Assets may be priced based on the preferences of this investor only*.

Next we consider a situation where markets are not necessarily complete and we will consider the conditions under which a representative agent exists whose preferences are such that asset prices are independent of the distribution of initial endowments.

(b) Preference Restrictions and the Representative Investor

Take investors who all have HARA preferences with potentially different constants. Investor *k* then has the following utility function:

$$(34)\qquad u_k(c_k) = [b/(b-1)](a_k + bc_k)^{1-(1/b)}.$$

We know from the Cass and Stiglitz (1970) result that two-fund separation holds, implying that markets are always effectively complete. Thus, a representative investor exists, as we saw in the previous subsection. We next show the stronger result that the asset prices derived from the preferences of the representative investor are independent of the distribution of initial endowments.

Find the preferences of the representative investor by construction. Maximize the expression in equation (28) subject to equation (29), using the utility function of equation (34). Taking γ as the Lagrangian multiplier for the aggregate resource constraint, we obtain the first-order condition for the consumption of investor *k*:

$$(35)\qquad (a_k + bc_k)^{-1/b} = \gamma/\lambda_i.$$

Solving for c_k and aggregating over all consumers yields:

$$(36)\qquad (\sum_{k=1}^{K} a_k) + bc = \gamma^{-b} \sum_{k=1}^{K} \lambda_k^b .$$

Equation (35) implies that:

$$(37)\qquad u_k(c_k) = (\lambda_k/\gamma)^{b-1}.$$

Thus, aggregate preferences would be given from equation (28) as:

$$(38) \qquad U[C(s)] = \gamma^{1-b} \sum_{k=1}^{K} \lambda_k^b \ .$$

Combine equations (36) and (38) to eliminate γ; this yields:

$$(39) \qquad U[C(s)] = [(\sum_{k=1}^{K} a_k) + bc(s)]^{1-(1/b)} \ [\sum_{k=1}^{K} \lambda_k^b]^{1/b} \ .$$

A similar derivation for utility at time *0* yields:

$$(40) \qquad U(C^0) = [(\sum_{k=1}^{K} a_k) + bc^0]^{1-(1/b)} \ [\sum_{k=1}^{K} \lambda_k^b]^{1/b} \ .$$

Now apply equation (33) by differentiating equations (39) and (40) to obtain:

$$(41) \qquad p_s = \beta \, \pi_s \, \frac{[(\sum_{k=1}^{K} a_k) + bc(s)]^{-(1/b)}}{[(\sum_{k=1}^{K} a_k) + bc^0]^{-(1/b)}} \ .$$

It follows that state prices do not depend on the distribution of initial endowments. It needs to be pointed out, though that both expectations were assumed to be homogeneous and that discount factors are equal across investors. In short:

> **RESULT 7 (REPRESENTATIVE INVESTOR WITH HARA PREFERENCES).** *If all investors have HARA preferences with discount factor and exponent equal across investors but with potentially different constants as given in equation (34), then a representative investor exists with HARA preferences. State prices and, hence, all asset prices as derived from the preferences of the representative investor are independent of the distribution of initial endowments across the investors.*

Note that HARA preferences are quite general and that most utility functions may be reasonably well approximated by some member of the HARA class. However, the assumption that the exponents in the HARA class are identical across all consumers is a strong one.

(c) Applications and Exercises

1. Huang and Litzenberger (1988) show that, when all investors have HARA preferences with identical exponents, but possibly different additive terms (the utility functions given in equation 34), that this is *necessary* and *sufficient* for sharing rules to be linear. That is, using equation (18) above:

$$c_i = f_i(C) = g_i + h_i C, \quad h_i > 0, \quad \text{for all } i.$$

Prove that these HARA preferences are sufficient for sharing rules to be linear, based on equation (16).

2. For the following statements, explain whether the statement is true, false, or uncertain.

(a) If markets are (actually and effectively) incomplete, a representative investor does not exist.
(b) If markets are (actually and effectively) incomplete, any Pareto inefficiencies may be eliminated by trades after states are realized.
(c) If markets are (actually and effectively) incomplete, it is incorrect to value investment opportunities by using expected net present value rules.