# Chapter II.  Mean-Variance Portfolio Choice

## 1. Introduction

Markowitz (1952, 1959) and Tobin (1958) were the first to consider portfolio choices in a mean-variance context. The current chapter is based on their work. The mean-variance approach produces optimal portfolio choices for individual investors, taking asset prices and payoff distributions as given. Building on the work of Markowitz and Tobin, Treynor (1961) and Sharpe (1963, 1964) independently constructed a "general equilibrium" model based on the mean-variance approach with given payoff distributions, but in which asset prices (or expected returns) become endogenous. Their model, as refined by Mossin (1966) and Lintner (1965, 1969), is now known as the Capital Asset Pricing Model, the CAPM. It is discussed in Chapter III. Before considering the asset pricing issues in Chapter III, we examine portfolio choice for individual investors in the current chapter.

## 2. The Mean-Variance Approach

Consider an individual investor/consumer in the context of a static model: decisions are made in the first period and the outcome occurs in the second period. We may think of this as a two-period model, but it is substantially different from models such as the Fisher two-period model discussed in the previous chapter, in which decisions are made in two periods. More appropriately we refer to the model as a one-period model in which decisions are made at the beginning of the period while the outcome occurs at the end of the period. As long as we distinguish decisions and outcomes, there is no need for time subscripts. The individual is assumed to maximize the expected utility of end-of-period consumption. Since life is assumed to end at the close of the period, there is no difference between consumption and wealth: all end-of-period wealth will be consumed. Taking a Taylor series approximation of the utility of consumption (or wealth) around its mean level yields:

$$(1) \qquad u(c) = u(\mu) + \frac{u'(\mu)(c-\mu)}{1!} + \frac{u''(\mu)(c-\mu)^2}{2!} + \frac{u'''(\mu)(c-\mu)^3}{3!} \ldots,$$

where $u(\ )$ represents a standard utility function, which is *monotonically increasing* and *concave* in its argument (indicating desirability and non-satiation as well as risk aversion), and $c$ represents stochastic end-of-period consumption, with mean of $\mu$.

Taking expectations on both sides of equation (1) produces:

$$(2) \qquad E[u(c)] = u(\mu) + \frac{u''(\mu)\sigma^2}{2!} + \frac{u'''(\mu)E[(c-\mu)^3]}{3!} \ldots .$$

This follows since $\sigma^2 \equiv E[(c-\mu)^2]$ is the variance of consumption around its mean. Given assumptions to be discussed in the following we can now write:

(3) $$v(\mu,\sigma) = E[u(c)], \quad \text{with } v_1(\mu,\sigma) > 0 \text{ and } v_2(\mu,\sigma) < 0.$$

Numerical subscripts indicate partial derivatives. Equation (3) holds under two different assumptions:

(i) *quadratic utility*. Under quadratic utility of *u*( ) the third and higher derivatives vanish and the second derivative becomes a constant. Assuming concavity produces a hump-shaped parabola. Quadratic utility thus unfortunately implies that the marginal utility of consumption becomes negative beyond a certain level $c = \bar{c}$; it must be assumed ex ante that consumption could never exceed $\bar{c}$. Additionally, it can be shown that the quadratic utility function implies Increasing Absolute Risk Aversion (IARA), which is not descriptive of typical consumers: It implies that *wealthier* consumers are *less* willing to gamble with a certain fixed amount of wealth, say \$1; thus risk is an inferior good under IARA. The signs of the partial derivatives of *v*( ) in equation (3) are easy to verify given the quadratic functional form assumed for *u*( ) and the fact that $c < \bar{c}$.

(ii) *elliptical distribution of c*. With an elliptical distribution of *c* all moments of *c* are fully characterized by its first two moments (mean and variance) only. So, the *n*th central moment of *c* will be a function of μ and $\sigma^2$ (or σ) only. Clearly, equation (3) holds under this assumption. [The partial derivatives can be signed given the fact that, for any elliptical distribution, an increase in σ is an example of a mean-preserving spread in the distribution of *c*; see also section 4].

Note that the normal distribution is a special case of an elliptical distribution. Other special cases include distributions with finite support. All elements of the class of elliptical distributions are symmetrical. There are also other distributions, like the uniform distribution, which may be characterized by mean and variance only, and for which equation (3) applies in the absence of portfolio formation. This fact may be responsible for Tobin's famous blunder. In the 1960s Tobin stated that mean-variance analysis of portfolios is valid for all distributions of individual assets' returns that are characterized by mean and variance only. His mistake was that random variables do not necessarily preserve their characterization under linear transformations, that is, distributions need not be "stable". This is crucial since portfolio returns are linear combinations of the returns on individual securities. The class of *elliptical distributions* consists of all distributions for which *linear combinations* of random variables are completely characterized by mean and variance, so that equation (3) applies.

The normal distribution has infinite support and, for that reason, is not descriptive of reality because of limited liability: in the context of the prevailing legal system, all stock holders are protected by limited liability so their rates of return are bounded below by -100%. Empirical distributions of returns, moreover, have tails that are leptokurtic, i.e., "fatter" than implied by the normal distribution. Additionally, many commonly used utility functions are not defined

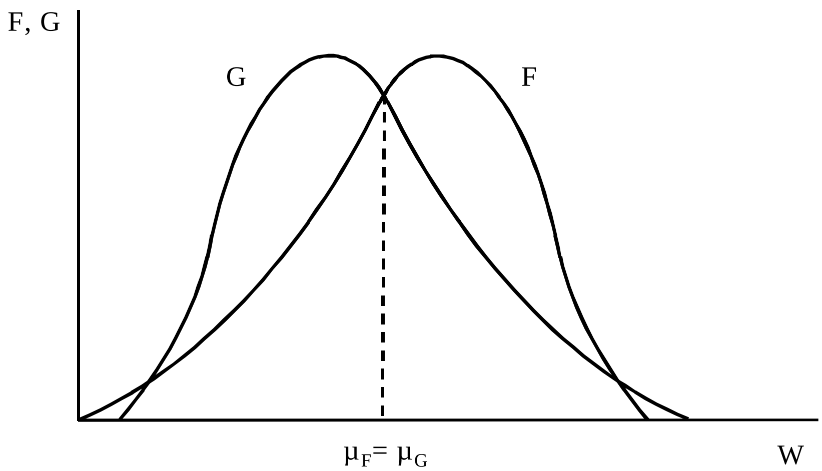


Figure 1
**Cost of Assuming Elliptical Distributions**
Distributions F and G have equal means and variances, but F is negatively skewed and G is positively skewed. Thus, G is typically preferred to F due to relatively less downside risk.

at negative values; thus, for instance, $E[\ln(x)]$ does not exist when $x$ is normally distributed, no matter what its mean and variance, since $x < 0$ is always possible. Favoring the use of the normal distribution is the Central Limit Theorem which states that the average or the sum of a large set of independent shocks approaches the normal distribution.

Thus, we can conduct mean-variance analysis of portfolio choices under assumption (i) or under assumption (ii). Alternatively, we could accept the mean-variance results as based on a second-order approximation. Further, Meyer (1987) has shown that comparative statics analysis is possible in a mean-variance framework for any utility function, as long as all possible final wealth distributions differ only by location and scale parameters. Under portfolio formation this condition is satisfied if the assets' returns are jointly elliptically distributed.

In the following we usually rely on assumption (ii) and, at times, more specifically assume a normal distribution. We can thus think of expected utility as a function of the mean and standard deviation (or variance) of consumption (or wealth) only. Figure 1 reveals the cost of making this assumption. Consider distributions *F* and *G*. They have equal mean and variance but *F* has negative skewness, while *G* has positive skewness. The assumption that only mean and variance of wealth matter implies that investors would be indifferent between these distributions. However, in reality, individuals seem to prefer *G* as it displays little downside risk but a lot of upside potential. [This is why people like to play the lottery even if their expected net payoff is negative; or why they pay an insurance premium to avoid distribution *F*. We may think of *F* and *G* differing by two steps: from *F* to a non-stochastic payoff at μ, for which an agent would pay the insurance premium, and then from a non-stochastic payoff at μ to *G*, for which the individual would pay a lottery mark-up].

We use the following notation for the remainder of this chapter as well as chapters III and IV:

$r_i$ = the return (sometimes called *rate* of return) on asset *i*.
$R_i$ = the *gross* return on asset *i* (that is, $1 + r_i$).
$n$ = number of available assets.
$r_f$ = the return on the risk free asset.
$\mu_i$ = mean of the return on asset *i*.
$\sigma_i$ = standard deviation of the return on asset *i*.
$\sigma_i^2$ = variance of the return on asset *i*.
$\sigma_{ij}$ = covariance between the returns on assets *i* and *j*.
$\rho_{ij}$ = correlation between the returns on assets *i* and *j*.
$s_i$ = the share of asset i in the portfolio.
$\mu_w$ = the mean level of end-of-period portfolio wealth.
$\sigma_w$ = the standard deviation of end-of-period portfolio wealth.
$\sigma_w^2$ = the variance of end-of-period portfolio wealth.
$\bar{w}$ = initial wealth.
$r \equiv r_p$ = the (rate of) return of a portfolio.
$\mu \equiv \mu_p$ = the mean of the portfolio return.
$\sigma \equiv \sigma_p$ = the standard deviation of the portfolio return.
$\sigma^2 \equiv \sigma_p^2$ = the variance of the portfolio return.

The following relations hold between some of the above variables (derive these yourself; if you have problems see Appendix):

$$\mu_w = \sum_{i=1}^{n} s_i (1 + \mu_i) \bar{w}$$

$$\sigma_w^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \sigma_{ij} \bar{w}^2$$

$$\mu_w = (1 + \mu) \bar{w}$$

$$\sigma_w = \sigma \bar{w}$$

$$\rho_{ij} = \sigma_{ij} / (\sigma_i \sigma_j)$$

$$\sum_{i=1}^{n} s_i = 1 \; .$$

Next we consider the opportunities for generating mean and variance of return on wealth. The same set of opportunities is assumed to be available to all consumer/investors [we will refer to *investors* rather than consumer/investors in the following].

## 3. THE PORTFOLIO FRONTIER

We discuss the opportunities available to investors for choosing different mean-variance of wealth combinations. To build intuition we first start with some simple examples and then increase the complexity.

*(a) One risky and one riskless asset*

The riskless asset is asset *0* (and may be indicated by a subscript *0* or subscript *f* ); the risky asset is asset *1*. The sum of the wealth shares invested in the assets must add to *1*:

(1) $$s_0 + s_1 = 1$$

The wealth at the end of the period is given as:

(2) $$w = (R_f s_0 + R_1 s_1)\,\bar{w}.$$

Taking expectations in equation (2) yields the mean level of wealth at the end of the period:

(3) $$\mu_w = [R_f s_0 + (1+\mu_1)\,s_1]\,\bar{w}\ .$$

Based on equation (2), the variance of end-of-period wealth can be derived as:

(4) $$\sigma_w^2 = \sigma_1^2 s_1^2 \bar{w}^2 \rightarrow s_1 = \sigma_w/(\sigma_1 \bar{w}).$$

[Note that you should be able to derive equation (4) based on equation (2) and the definition of variance]. The last part of (4) implies that choosing the share of the risky asset is tantamount to choosing the standard deviation of wealth.

Use equations (1) and (4) to eliminate the portfolio shares $s_i$ :

(5) $$\mu_w = (1+r_f)\,\bar{w} + (\sigma_w/\sigma_1)(\mu_1 - r_f).$$

Equation (5) provides an equation describing the opportunities for the investor for obtaining an average end-of-period wealth level as related to the variance of wealth that the investor is willing to accept. Note that the ability for the investor to choose portfolio composition has been translated (see equation 4) into the ability of the investor to choose a "risk level", the standard deviation of wealth. Equation (5) shows that the choice of risk (standard deviation of wealth) determines the mean level of wealth that can be obtained.

The opportunities available described in equation (5) are investor specific: even if all investors have access to the same assets, the mean level of end-of-period portfolio wealth still differs by individual due to the presence of initial wealth $\bar{w}$ in equation (5). In order to obtain general portfolio choice results and examine general equilibrium implications it is important, however, to look at opportunity sets that do not vary by investor. For this reason we consider mean and variance of portfolio return rather than mean and variance of portfolio wealth. In terms of portfolio returns, equation (5) can be rewritten as:

$$\text{(6)} \qquad \mu = r_f + (\sigma/\sigma_1)(\mu_1 - r_f).$$

[Make sure to check this derivation]. Initial wealth no longer appears in equation (6). Every investor has equal opportunity to choose a particular (μ, σ) combination.

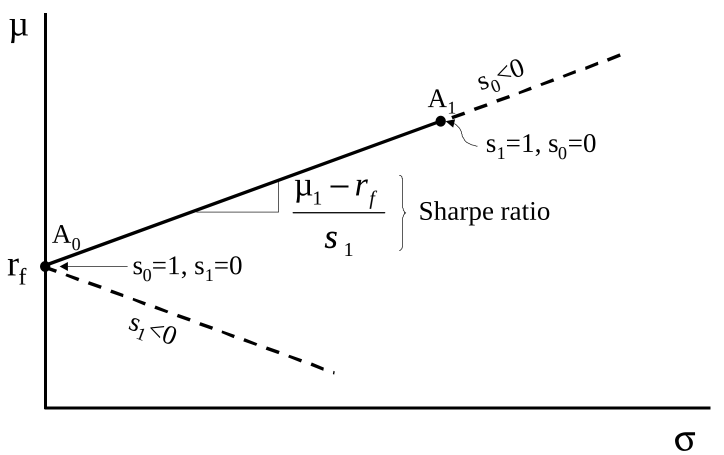


Figure 2
**Opportunity Set for One Risky and One Riskless Asset**
$A_1$ and $A_0$ represent risky and riskless assets, respectively. Portfolio shares ($s_0$, $s_1$) range from zero to one along the solid portion of the opportunity set. Short-sales of either asset extend the opportunity set along the dashed lines.

Figure 2 shows the opportunity set available to any investor. The dotted line indicates opportunities that are only possible if short-sales are allowed ($s_0 < 0$ or $s_1 < 0$). The intercept of the opportunity line is at the risk free rate, since the standard deviation of portfolio return is only zero when the whole portfolio is in the risk free asset ($s_0 = 1$; $s_1 = 0$). Note that the standard deviation cannot fall below zero. If short sales occur at this point (short-selling the risky asset and investing the proceeds in the riskless asset), positive (not negative) risk occurs due to the investor's obligation in servicing the risky asset; this explains the kink in the opportunity line. The slope of the opportunity line is given as:

$$\text{(7)} \qquad \frac{d\mu}{d\sigma} = \frac{\mu_1 - r_f}{\sigma_1}.$$

This slope is positive under the reasonable assumption (required in general equilibrium) that the mean return on the risky asset exceeds the return on the risk free asset. This slope indicates the "price of risk reduction" that each investor faces. It shows by how much expected portfolio return rises if the standard deviation (chosen by the investor) increases by one unit. The price of risk reduction derived here is generally referred to as the *Sharpe Ratio* (named after William Sharpe, Nobel prize winner, and one of the inventors of the CAPM).

*(b) Two risky assets*

Notation-wise assume that a risky asset *2* is added and that the risk free asset, asset *0*, is not available. Consider as before the portfolio return (rather than portfolio wealth), which is given as:

$$(8) \qquad r = r_1 s_1 + r_2 s_2 .$$

As before the portfolio shares need to add up to one:

$$(9) \qquad s_1 + s_2 = 1$$

Taking expectations yields the mean portfolio return:

$$(10) \qquad \mu = \mu_1 s_1 + \mu_2 s_2 .$$

Based on equation (8), portfolio variance is [make sure to derive this yourself]:

$$(11) \qquad \sigma^2 = \sigma_1^2 s_1^2 + 2\rho_{12}\sigma_1\sigma_2 s_1 s_2 + \sigma_2^2 s_2^2 .$$

Combining equations (10) and (11), and using (9) to eliminate the portfolio shares, provides the feasible combinations of mean and variance. The relation between mean and variance for a set of risky assets is named the *portfolio frontier* if it associates with every mean return the lowest possible variance of return available to the investor. In the case of two risky assets, the lowest possible variance happens to be equal to the one value of the variance feasible for a given mean. Combining equations (10), (11), and (9) yields the portfolio frontier for two risky assets:

$$(12) \qquad \sigma^2 = \frac{1}{(\mu_2 - \mu_1)^2}\left[\sigma_1^2(\mu - \mu_2)^2 - 2\rho_{12}\sigma_1\sigma_2(\mu - \mu_1)(\mu - \mu_2) + \sigma_2^2(\mu - \mu_1)^2\right].$$

Note that equation (12) represents a parabola in mean-variance space, but a hyperbola in mean-standard deviation space. Figure 3 displays the portfolio frontier in mean-standard deviation space for different values of the correlation between the two risky assets.

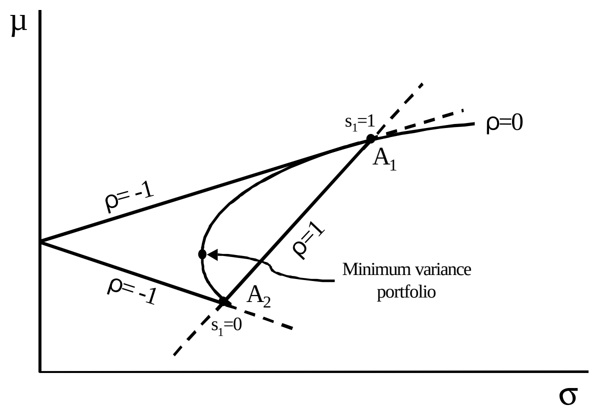


Figure 3
**Portfolio Frontiers for Two Risky Assets**
The shape of the portfolio frontier for two risky assets, $A_1$ and $A_2$, depends on the degree of correlation between the two assets.

In a recent interview (Revisiting The Capital Asset Pricing Model by Jonathan Burton, Dow Jones Asset Manager, May/June 1998, pp. 20-28), William Sharpe said: "Investment texts in the pre- Markowitz era were simplistic: Don't put all your eggs in one basket, or put them in a basket and watch it closely. There was little quantification. To this day, people recommend a compartmentalized approach. You have one pot for your college fund, another for your retirement fund, another for your unemployment fund. People's tendencies when they deal with these issues often lead to suboptimal solutions because they don't take covariance into account. Correlation is important. You want to think about how things move together."

As Sharpe indicates, the correlation between the risky assets is a crucial aspect of any portfolio decision, so to get an idea of the general shapes of the portfolio frontier that are possible, we explicitly consider several extreme assumptions about the correlation $\rho_{12} \equiv \rho$ between the returns of assets *1* and *2*.

*(i)* $\rho = 1$

The assets are perfectly correlated. Equation (11) now simplifies to:

$$\text{(11a)} \qquad \sigma = \sigma_1 s_1 + \sigma_2 s_2 .$$

Figure 3 shows that the portfolio frontier becomes a straight line sloping up from the point where $s_1 = 0$ to the point where $s_1 = 1$. The dotted lines indicate again the opportunities when short sales are permitted. When the returns on the risky assets are perfectly correlated, no diversification benefits occur and combining the assets will just lead to a linear combination between the extreme positions of putting the whole portfolio in a single asset.

*(ii)* $\rho = -1$

The assets are perfectly negatively correlated. Equation (11) becomes:

$$\text{(11b)} \qquad \sigma = |\sigma_1 s_1 - \sigma_2 s_2|.$$

Note that, strictly speaking, the absolute value should also be taken in equation (11a) if short sales are allowed. Diversification benefits are maximal due to the negative correlation between the asset returns. Figure 3, shows that the portfolio can be diversified such that no risk is incurred; asset *2* can be a perfect hedge for asset *1* and vice versa.

*(iii)* $\rho = 0$

The assets are uncorrelated. Some may think that this implies that diversification is not possible; in fact, the benefits of diversification are quite clear in this case. It is one of the basic insights necessary to understand portfolio choice. Even though the middle term in equation (11) drops out due to this assumption, the analysis in this case is substantially more complex than in the previous two cases. Equation (11) becomes:

$$\text{(11c)} \qquad \sigma = (\sigma_1^2 s_1^2 + \sigma_2^2 s_2^2)^{1/2}.$$

Consider the mean/standard deviation tradeoff in this case derived from equations (10) and (11c):

$$\text{(13)} \qquad \frac{d\mu}{d\sigma} = \frac{d\mu/ds_1}{d\sigma/ds_1} = \frac{\mu_1 - \mu_2}{(s_1\sigma_1^2 - s_2\sigma_2^2)/\sigma}.$$

If we assume that $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$, the sign of the slope in equation (13) depends on the denominator. It is easy to see that at some point the slope is vertical. The portfolio that produces this point is called the *minimum variance portfolio*. Further, at the fully undiversified point where $s_1 = 0$, the slope must be negative as shown in Figure 3. Thus, starting from this undiversified point, more diversification is beneficial for every investor with mean-variance preferences: mean return rises while standard deviation falls.

It is clear that the portfolio frontier in case (iii) lies between the frontiers of cases (i) and (ii). It can be shown that this is true for the general case as well. For general correlation between assets 1 and 2, it is also true that the portfolio frontier has the same hyperbolic shape as in case (iii).

*(c) An arbitrary number of risky assets*

We assume here that investors may invest in a total of *n* risky assets and that no riskless asset exists. Short sales are not restricted. The portfolio frontier in this case was rigorously derived by Merton (1972). Figure 4 shows the frontier. Each dot represents one asset's $(\mu_i, \sigma_i)$ characterization. Diversification between any two assets (dots in

Figure 4) will produce a hyperbola as discussed above. But any point on the hyperbola of two assets can also be considered as one asset and can be combined with another asset to yield a new hyperbola. Combining all pairs of assets in this manner, in the limit will yield an enveloping hyperbola that lies to the left of any individual dot.

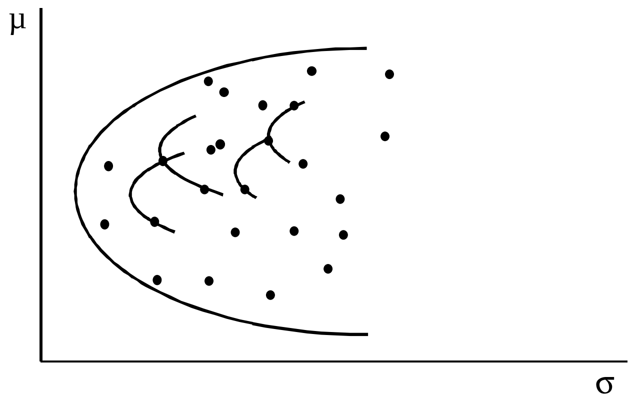


Figure 4
**Portfolio Frontier for n-Risky Assets**
For an arbitrary number of risky assets, combining all pairs of risky assets produces an enveloping hyperbola that lies to the left of any individual set of assets.

Mathematically the portfolio frontier can be found by minimizing σ subject to a given μ. The dual of this decision problem does not provide the same solution: maximizing μ subject to a given σ only produces the upper half of the portfolio frontier. The lower half is dominated since a higher μ can be found for any feasible σ. The upper half of the portfolio frontier obtained in this manner is called the *efficient frontier* for obvious reasons. Empirically, if the assumptions leading to mean-variance analysis are justified, we expect that no individual's complete portfolio lies below the efficient frontier.

*Formal derivation of the portfolio frontier*

Consider the following variable definitions:

$\mathbf{\Sigma} = [\sigma_{ij}]$, represents the *n x n* variance-covariance matrix of the *n* asset returns, where $\sigma_{ii} \equiv \sigma_i^2$.
$\boldsymbol{\mu}$ is a *1 x n* column vector of the expected returns $\mu_i$ .
**s** represents a *1 x n* column vector of the portfolio shares $s_i$.
$\mathbf{x}^\mathbf{T}$ represents the transpose of vector **x**.
**1** represents a *1 x n* column vector of 1's
$\mathbf{\Sigma}^{-1}$ represents the inverse of matrix $\mathbf{\Sigma}$

The portfolio frontier is found by minimizing portfolio variance subject to a given portfolio mean:

(14) Minimize with respect to $\boldsymbol{s}$: $\frac{1}{2}\, \boldsymbol{s}^{T} \boldsymbol{\Sigma} \boldsymbol{s}$

(15) Subject to: $\boldsymbol{\mu}^{T} \boldsymbol{s} = \mu_p$

(16) $\mathbf{1}^{T} \boldsymbol{s} = 1$

Thus portfolio variance is minimized subject to a given expected portfolio return $\mu_p$ and given that all portfolio shares add up to 1.

Using the Lagrangian method with mulipliers $\lambda$ and $\kappa$ for constraints (15) and (16), respectively, produces the following first-order condition:

(17) $\boldsymbol{s}^{T} \boldsymbol{\Sigma} - \lambda\, \boldsymbol{\mu}^{T} - \kappa\, \mathbf{1}^{T} = 0.$

Solving for the optimal portfolio shares yields:

(18) $\boldsymbol{s}^{T*} = \lambda\, \boldsymbol{\mu}^{T} \boldsymbol{\Sigma}^{-1} + \kappa\, \mathbf{1}^{T} \boldsymbol{\Sigma}^{-1}$

Based on the fact that $\boldsymbol{\Sigma}$ is positive definite we can conclude that $\boldsymbol{s}^{T*}$ does minimize the variance and that the solution obtained here for the portfolio shares is unique. Nothing, however, in this partial-equilibrium framework guarantees that all portfolio shares are positive or below one. Note that invertibility of the variance-covariance matrix requires that no risk-free asset is included and that no two assets are perfect substitutes; either of which would cause $\boldsymbol{\Sigma}$ to become singular. Furthermore, the assumed existence of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^{-1}$ presupposes that all variances are finite (which, empirically, is debatable due to the excess kurtosis in observed asset returns).

Post-multiplying equation (17) by $\mathbf{s}$ and using constraints (15) and (16) gives:

(19) $\sigma_p^2 = \lambda \mu_p + \kappa.$

Post-multiplying equation (18) by $\boldsymbol{\mu}$ and separately by $\mathbf{1}$ yields the following two equations:

(20) $\mu_p = \lambda\, \boldsymbol{\mu}^{T} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \kappa\, \mathbf{1}^{T} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$

(21) $1 = \lambda\, \boldsymbol{\mu}^{T} \boldsymbol{\Sigma}^{-1} \mathbf{1} + \kappa\, \mathbf{1}^{T} \boldsymbol{\Sigma}^{-1} \mathbf{1}$ .

Define:

(22) $A = \boldsymbol{\mu}^{T} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, $B = \mathbf{1}^{T} \boldsymbol{\Sigma}^{-1} \mathbf{1}$, $C = \mathbf{1}^{T} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^{T} \boldsymbol{\Sigma}^{-1} \mathbf{1}$, and $D = AB - C^2$.

Note that A,B, C, and D are scalars that depend only on the constant parameters of the set of available assets. It is now straightforward to solve for $\lambda$ and $\kappa$ from equations (20) and (21):

(23) $\lambda = (B\mu_p - C)/D$

(24) $\kappa = (A - C\mu_p)/D$ .

Plugging (23) and (24) into equation (19) yields an explicit expression of the portfolio frontier:

(25) $\sigma_p^2 = (B\mu_p^2 - 2C\mu_p + A)/D$ .

The portfolio frontier is a hyperbola in mean-standard deviation space as in case (*b*) with two risky assets. The reason is intuitive: any two points on the frontier can be thought of as mutual funds that are individual assets. Taking different combinations of these two assets must trace out a hyperbola based on case (*b*); but there is no way that this hyperbola can be different from the *n*-asset frontier as it can, at no point, lie to the left of the *n*-asset frontier (or the frontier wouldn't be a true frontier). Thus, once we understand case (*b*), we can deduce logically that the *n*-asset frontier must be a hyperbola as well. We further can easily deduce the important result (which we have not formally proven) that the *n*-asset frontier can be traced out by combining any two mutual funds that lie on the frontier.

Based on equation (25) it is is easy to find the *minimum variance* obtainable without a riskless asset. Differentiating with respect to $\mu_p$ and setting equal to zero yields $\mu_p = C/B$ so that we obtain $\sigma_p^2 = 1/B$ (after using the definition of $D$ in (22).

(*d*) *An arbitrary number of risky assets plus a riskless asset*

The introduction of the riskless asset substantially simplifies the analysis. While case (*c*) becomes quite similar to case (*b*); case (*d*) is similar to case (*a*). Intuitively consider the *n*-asset frontier of case (*c*) providing the investment opportunities given availability of risky assets only. Clearly, opportunities increase if an additional asset is introduced that lies outside of the *n*-asset frontier. The "dot" indicating for each basic asset *i* the ($\mu_i$, $\sigma_i$) combination, in the case of the risk free asset becomes ($r_f$ , *0*). In Figure 5 investment opportunities can clearly be extended substantially by taking, say, convex combinations of the minimum variance portfolio and the risk free asset (the different convex combinations form a straight line as in case (*a*)). Investment opportunities are extended further if the risk free asset is combined with an efficient portfolio, such as *A* in Figure 5; the additional available investment opportunities are indicated by the dashed line (and any area that lies below it). What is the furthest that the opportunity set can be extended? Graphically, it is obvious that the straight line extending from the risk free asset to the portfolio frontier should be pushed up as far as possible. The true opportunity frontier is obtained when the straight line is just tangent to the portfolio frontier at point *T* in Figure 5.

The opportunity frontier as determined by the straight line going through the risk free rate on the vertical axis and the tangency point on the portfolio frontier is called the *Capital Market Line* (or, sometimes, the *Portfolio Market Line*). Call the unique portfolio of risky assets that reaches the portfolio frontier at point *T* the *tangency portfolio.* Then point *T* on the CML implies full investment of all wealth in the tangency portfolio and no investment in the riskless asset. Any point on the CML to the right of *T* implies borrowing (going short) on the riskless asset. Any point on the CML below $r_f$ implies going short on the tangency portfolio and investing the proceeds plus initial wealth in the risk

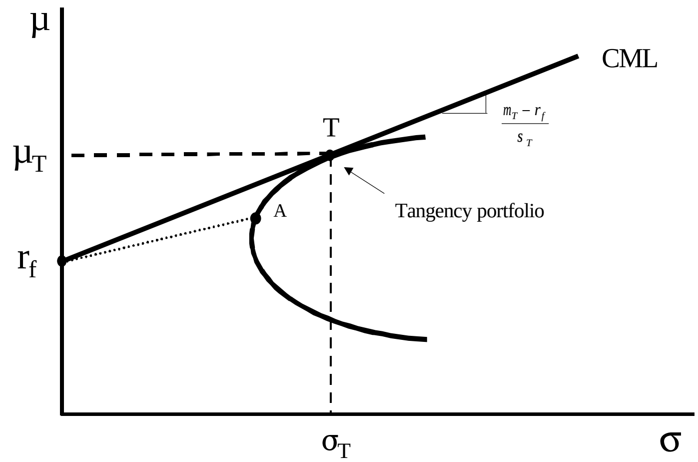


Figure 5
**Capital Market Line**
The existence of a risk-free asset extends the opportunity set. The opportunity frontier is the Capital Market Line (CML). The tangency portfolio is the unique portfolio of risky assets that reaches the portfolio frontier

free asset. As before, the CML here displays a kink since standard deviation cannot become negative: any further holdings of the
risk free asset beyond initial wealth increase risk, even though expect portfolio return still falls. Clearly, the part below the kink is irrelevant and does not affect investment opportunities. The equation for the CML can easily be constructed as:

$$(26) \qquad \mu_p = r_f + \left( \frac{\mu_T - r_f}{\sigma_T} \right) \sigma_p .$$

The slope in equation (26) is the Sharpe ratio or the "price of risk reduction".

All investment opportunities for any risk averse investor are summarized by the CML. But the opportunities on the CML can be generated by two "portfolios" only: the tangency portfolio and the risk free asset. Given our basic assumptions, every investor will be on the CML. To get there, she must split her wealth over the tangency portfolio and the risk free asset. Thus, every investor holds risky assets in the same proportion, as defined by the tangency portfolio. The result that all investor opportunities are summarized by a fixed set of portfolios or "mutual funds" is a *portfolio separation* result often referred to as a Mutual Fund Theorem. In this case, the investor will hold two mutual funds (tangency portfolio and risk free asset), although only one risky fund. The result that holding only one risky mutual fund (such as the tangency portfolio) and a risky asset is sufficient for all investment purposes is known by various names: two-fund separation, Tobin's separation theorem, a two-fund mutual fund theorem, or a one-risky-fund mutual fund theorem. Often this result is called The Mutual Fund Theorem as it is the most prominent one in its class.

The price of risk reduction is the same for each individual, and equals the Sharpe ratio $(\mu_T - r_f)/\sigma_T$, no matter how risk averse the individual is! This result is akin to the Fisher separation result discussed in Chapter I. Investors differ only in terms of which fraction of their wealth they put in the riskless asset. A more explicit discussion of preferences and portfolio choice in the mean-variance model follows but first we consider the formal derivation of the CML.

*Formal derivation of the Capital Market Line*

The formal derivation of the CML is based on efficient portfolio choice for the extended opportunity set when a risk free asset is added to the set of risky assets.

(27) Minimize with respect to $\boldsymbol{s}$: $\frac{1}{2}\, \boldsymbol{s}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{s}$ .

(28) Subject to: $\boldsymbol{\mu}^{\boldsymbol{T}} \boldsymbol{s} + r_f (1 - \mathbf{1}^{\boldsymbol{T}} \boldsymbol{s}) = \mu_p$ .

Portfolio variance is minimized subject to a given expected portfolio return which includes a fraction of wealth invested in the riskless asset. Equation (28) can be simplified by using the definition of the *expected excess return*:

(29) $e_i = \mu_i - r_f$ ; $\boldsymbol{e} = \boldsymbol{\mu} - \boldsymbol{r_f} \mathbf{1}$.

Thus, we can rewrite equation (28) as:

(28a) Subject to: $\boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{s} = e_p$ .

Using the Lagrangian method with muliplier $\lambda$ constraint (28a) produces the following first-order condition:

(30) $\boldsymbol{s}^{\boldsymbol{T}} \boldsymbol{\Sigma} - \lambda\, \boldsymbol{e}^{\boldsymbol{T}} = 0.$

Solving for the optimal portfolio shares when a risk free asset exists yields:

(31) $\boldsymbol{s}^{\boldsymbol{T}*} = \lambda\, \boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{\Sigma}^{-\mathbf{1}}$.

The intuition for this expression will be discussed in the next subsection. First we derive the CML. Post-multiplying equation (31) by the vector of expected excess returns produces:

(32) $e_{\mathrm{p}} = \lambda\, \boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{\Sigma}^{-\mathbf{1}} \boldsymbol{e} \equiv \lambda F$ ,

where scalar $F > 0$. Post-multiplying equation (30) by the vector of portfolio shares yields:

(33) $\sigma_p^2 = \lambda e_p$ .

The CML is then given by equations (32) and (33) after eliminating $\lambda$ :

$$(34) \qquad e_p^2 = F\sigma_p^2 \rightarrow e_p = \pm F^{1/2}\sigma_p \ ,$$

where we can take the positive branch of the two line segments as it dominates the negative branch. Note that the positive branch can be found graphically by drawing the tangency with the portfolio frontier for the case with risky assets only, if the risk free rate is below the mean return associated with the minimum variance portfolio. [It can be shown that, in the case that can be ruled out in general equilibrium where risk free rate is above the mean return associated with the minimum variance portfolio, the lower line segment is tangent to the portfolio frontier.]

*(e) Applications and Exercises*

The following exercises deal with various issues related to the portfolio frontier.

1. Only two risky assets exist with mean and standard deviations of the return given as $\mu_i$ and $\sigma_i$ for i=1,2.

   (a) Algebraically obtain the portfolio frontier.

   Now add a riskless asset with return $r_0 = 1$ and set $\sigma_1 = \sigma_2 = 1$, $\sigma_{12} = 0$ and $\mu_1 = 1$, $\mu_2 = 2$.

   (b) Obtain the portfolio frontier (for risky assets) for these values and find the minimum variance portfolio.
   (c) What are the composition and the mean and standard deviation of the tangency portfolio?

2. A Wall Street Journal article of February 10, 1997 reports on a practical risk correction procedure employed by Morgan Stanley and devised by Nobel laureate Franco Modigliani and his grand-daughter Leah Modigliani (a stock analyst at Morgan Stanley). To apply the procedure "... Morgan Stanley tweaks a fund's portfolio until the volatility exactly equals that of a benchmark like Standard & Poor's 500-stock index. To do so, they either increase or decrease exposure to stocks. Once that is done, the yield on the new hypothetical portfolio equals the risk-adjusted return."

   (a) Describe the correction proposed by the Modiglianis to account for differences in risk between mutual funds. Use a graph in mean-standard deviation space to support your explanation.
   (b) Explain why the Modiglianis' risk correction would be inappropriate (i) if a risk free asset does not exist, or (ii) if investors choose to hold more than one mutual fund.

3. Consider an opportunity set consisting of two risky assets. These assets have identical means $\mu_1 = \mu_2 = 1$, variances of $\sigma_1^2 = 2$ and $\sigma_2^2 = 4$, and covariance of $\sigma_{12} = 1$.

   (a) Calculate the minimum-variance portfolio and obtain the “efficient frontier.” Carefully draw the efficient frontier and indicate the individual assets in mean-standard deviation space. (Hint: note that the frontier may be a little unusual since both assets have the same mean return).

   (b) Suppose that the risk free rate equals $r_0 = ½$. Find the expression for the Capital Market Line.

4. Graphically draw the Capital Market Line and the portfolio frontier in mean-*variance* space.

5. For the case with *n* risky assets and a risk free asset, derive that the portfolio share of the riskless asset is equal to:

$$s_f = \frac{\boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{\Sigma}^{-\mathbf{1}} (\boldsymbol{e} - e^p \mathbf{1})}{\boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{\Sigma}^{-\mathbf{1}} \boldsymbol{e}}.$$

## 4. PREFERENCES AND PORTFOLIO CHOICE

*(a) Indifference Curves*

We discuss the properties of the indifference curves in mean-standard deviation space under the assumption that returns are elliptically distributed. Equation 2.3 can be written as:

$$v(\mu, \sigma) = E\{u[\bar{w}(1 + r)]\}, \tag{1}$$

where $w = \bar{w}(1 + r)$ and $r$ is the elliptically distributed portfolio return. Any random variable $r$ with arbitrary mean and variance can be written as $r = \mu + \sigma\varepsilon$, where $\varepsilon$ is a standardized random variable with mean of zero and variance equal to one. When $r$ is elliptically distributed, $\varepsilon$ will be elliptically distributed and independent of $\mu$ and $\sigma$. We have:

$$v(\mu, \sigma) = E\{u[\bar{w}(1 + \mu + \sigma\varepsilon)]\}. \tag{2}$$

It is now straightforward to conduct comparative statics analysis:

$$v_\mu(\mu, \sigma) = \bar{w}E[u'(w)] > 0 \tag{3}$$

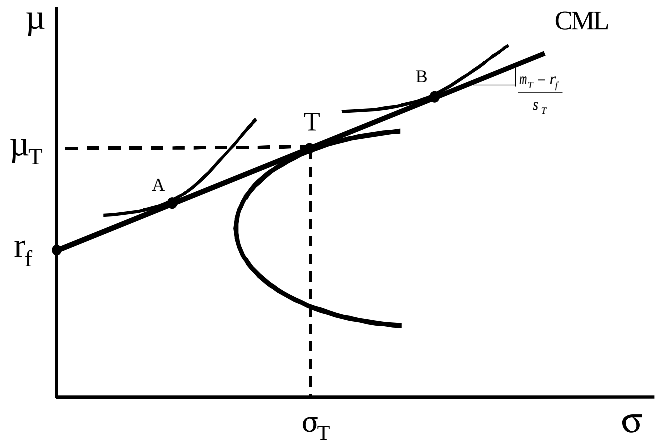


Figure 6
**Portfolio Choice**
The more risk-averse investor divides his wealth between the tangency portfolio and the risk-free asset and chooses a portfolio such as A. The less risk-averse investor may short sell the risk-free asset and choose a portfolio such as B.

$$(4) \qquad v_\sigma(\mu,\sigma) = \bar{w}E[\varepsilon u'(w)] = \bar{w}\,Cov[\varepsilon, u'(w)] < 0.$$

Here subscripts indicate partial derivatives. The positive sign in equation (3) follows since marginal utility is always positive. The second equality in equation (4) holds from the definition of covariance [see Appendix] plus the fact that E(ε) = 0. The negative sign in equation (4) then follows because, due to the concavity of the utility function, *w* falls in ε everywhere so that the covariance is negative.

We define indifference curves for a particular utility level $\bar{v}$ as:

$$(5) \qquad v(\mu,\sigma) = \bar{v}.$$

Totally differentiating equation (5) yields the slope of the indifference curve as:

$$(6) \qquad \frac{d\mu}{d\sigma} = \frac{-v_\sigma(\mu,\sigma)}{v_\mu(\mu,\sigma)} > 0.$$

It is also clear that utility rises in the north-west direction, so higher indifference curves lie further to the left (lower σ) and above (higher μ). Consider now whether the shape of these indifference curves is concave or convex. Differentiating (6) yields:

$$(7) \qquad \frac{d^2\mu}{d\sigma^2} = \frac{-v_{\sigma\sigma}v_\mu + v_{\mu\mu}v_\sigma}{v_\mu^2} > 0.$$

This follows by differentiating (3) and (4):

(8) $$v_{\mu\mu} = \bar{w}^2 E[u''(w)] < 0$$

(9) $$v_{\sigma\sigma} = \bar{w}^2 E[\varepsilon^2 u''(w)] < 0.$$

The signs in (8) and (9) follow from the concavity of the utility function.

*(b) Portfolio Choice*

We examine the portfolio choices given the availability of *n* risky assets and a risk free asset and assumptions that guarantee mean-variance preferences. Figure 6 shows the optimum for a typical investor. It is based on the opportunity set derived in section 3(d) and the indifference curves discussed above. Clearly every investor will end up on the CML and will thus divide his wealth between the tangency portfolio and the risk free asset. The composition of risky-asset portfolios will be identical for all investors. The fraction of wealth put in the risk free asset will generally differ by investor.

From equation (2) it should be clear that, even when two investors have identical preferences, choices may differ depending on initial wealth $\bar{w}$. In equilibrium, every investor faces the same "price of risk reduction" along the CML. Clearly this is the case even though investors may have inherently different degrees of risk aversion. To identify different degrees of risk aversion, one may compare the slopes of the indifference curves at a given mean-standard deviation reference point. Steeper indifference curves indicate more risk aversion. An investor with more risk aversion will eventually still face the same risk-return tradeoff but will do so at a higher mean return and higher standard deviation, which implies a larger fraction of wealth invested in the tangency portfolio of risky assets.

***** *(c) Intuition for the optimal portfolio shares equation*

For the case with a risk free asset, equation (3.31) above provides a simple expression for the efficient portfolio shares of a typical investor:

(10) $$\boldsymbol{s}^{\boldsymbol{T}*} = \lambda\, \boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{\Sigma}^{-1}.$$

For given $\lambda$, efficient choice of shares is, intuitively, proportional to expected excess returns but also depend, less intuitively, on the inverse variance-covariance matrix of all risky assets. Stevens (1998) provides intuition related to this inverse matrix. Stevens derives that the typical *element $c_{ij}$ of the inverse variance-covariance matrix* can be written as follows:

(11) $$c_{ij} = \frac{-\beta_{ij}}{\sigma_i^2(1 - R_i^2)},$$

where $\beta_{ij}$ represents the slope coefficient for return *j* of a multiple regression which regresses the return of asset *i* on the returns of all *n-1* risky assets (all risky assets but excluding asset *i*). (We can set $\beta_{ii}$ = -1 as occurs when the left-hand side of equation (12) below is moved to the right-hand side). $R_i^2$ represents the coefficient of determination of this regression:

$$(12) \qquad r_i - r_f = \beta_{i0} + \beta_{i1}(r_1 - r_f) + \ldots + \beta_{i,i-1}(r_{i-1} - r_f) + \beta_{i,i+1}(r_{i+1} - r_f) + \ldots + \beta_{in}(r_n - r_f) + \varepsilon_i$$

Now consider first a case where all risky assets have independent returns. The off-diagonal elements then are *0* and the diagonal elements equal $1/\sigma_i^2$. Portfolio shares (from equation (10), summing the elements of the *i*th column of the inverse variance-covariance matrix weighted by the excess returns for each row asset) thus are proportional to the ratio of the asset's mean excess return and its variance, $e_i/\sigma_i^2$.

Similarly consider the general case. Summing the column elements weighted by excess mean returns yields as the denominator $\sigma_i^2(1 - R_i^2)$. The denominator can be interpreted as the undiversifiable part of the risk of asset *i* (the variance of asset *i*'s return multiplied by one minus the fraction of the asset's return explainable by the returns on all other assets–or the variance of ε in equation (12)). An investor could obtain this variance as a portfolio variance by creating a "hedging portfolio", holding, for every asset *j*, $\beta_{ij}$ dollars of asset *j* to each dollar in asset *i* . The numerator for the portfolio weight expression for asset *i* becomes: $e_i - \Sigma_{j \neq i} \beta_{ij} e_j$. This represents the expected excess return on the hedging portfolio; it is also equal to the intercept $\beta_{i0}$ in equation (12). Thus portfolio shares can be given as:

$$(13) \qquad s_i = \lambda \left( \frac{e_i - \Sigma_{j \neq i} \beta_{ij} e_j}{\sigma_i^2 (1 - R_i^2)} \right) = \frac{\lambda \beta_{i0}}{\sigma_i^2 (1 - R_i^2)} .$$

Portfolio shares are proportional to the ratio of the expected excess return from a hedging portfolio designed to minimize residual variance of asset *i*, and this residual variance.

*(d) Applications and Exercises*

1. Suppose you purchase a house. Would it ever be mean-variance efficient to use all of your stock market portfolio to pay off part of your mortgage?

Assume that you choose the value of the house *H*, the value of a stock-index portfolio *S*, and the size of your mortgage *M*, subject to your initial wealth $\bar{w}$. You can ignore all other assets and can assume that the mortgage interest is equal to the risk free rate. Further, the mortgage loan cannot be negative. It is assumed that you consider house, stock portfolio, and mortgage as pure financial assets (or a liability as in the case of the mortgage) and care only about the mean return on and standard deviation of your initial wealth. The mean return on the house and the stock-index both exceed the mortgage rate.

*[Hints. The question can be summarized as "Would an efficient portfolio ever have S=0?" Make sure to first formulate the budget constraint. You may consider defining investment shares in the three assets so that these shares add up to 1. The decision problem will be to minimize variance of return on wealth subject to a feasible mean return on wealth, so find expressions for the variance of return on wealth and the mean return on wealth first. It will be helpful to draw portfolio frontier and capital market line, keeping in mind that it is the capital market line that provides the set of mean-variance efficient portfolios].*

***** 2. Balvers and Mitchell (2000) consider a dynamic portfolio choice problem that can be captured in a static framework. Suppose an investor maximizes the utility of end-of-period *T* wealth. Assume that the investor must choose (irrevocably) at time *0* the portfolio allocation between investment in a risky mutual fund and in a riskless asset (no other choices are available) for each of the following *T* periods. Assume that the risk free return is constant over time (not required) and that the risky mutual fund's return follows the following moving average process: $r_t = \bar{r} - \delta\varepsilon_{t-1} + \varepsilon_t$, where subscript *t* indicates the time period and $\varepsilon_t$ has the elliptical distribution.

(a) Explain why this dynamic 2-asset problem for *T* periods is similar to a static *T* risky asset plus a riskless asset problem and that the key difference is that in the dynamic case the variance-covariance matrix is given specifically.

(b) Using the results in Stevens (1998), provide the solution for optimal investment in the risky mutual fund over time.