# Reducing the Dimensionality of Linear Quadratic Control Problems*

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**ABSTRACT**

In linear-quadratic control (LQC) problems with some singularities in the control cost, the state cost, and/or the transition matrices, we derive a reduction of the dimension of the Riccati matrix, simplifying iteration and solution. Employing a novel transformation, we show that, under a certain rank condition, the matrix of optimal feedback coefficients is linear in the reduced Riccati matrix. For a substantive class of problems, our technique permits scalar iteration, leading to simple analytical solution.

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# Reducing the Dimensionality of Linear Quadratic Control Problems

## 1. Introduction

The preeminence of computable general equilibrium models has stimulated interest in the solution procedures for larger-scale models. Most commonly, linear rational expectations models are considered which, typically, are derivable from linear-quadratic control (LQC) problems. The recent work by Sims (2000), Binder and Pesaran (1997, 2000), King and Watson (1998), Amman (1997), Amman and Neudecker (1997), Anderson et al. (1997), Anderson and Moore (1985), Ehlgen (1999), and Klein (2000) concentrates on numerical procedures that (1) allow speedy and accurate computation of results, and (2) apply as generally as possible, in particular to systems with non-invertibilities stemming from a singular transition matrix or a singular control cost matrix. These papers improve on the work of Vaughan (1970) and Blanchard and Kahn (1980).[1]

The purpose of this paper is twofold. First, we show that when the control cost matrix or the transition matrix is singular, the dimension of the Riccati equation can be reduced, allowing existing solution techniques or direct iteration for the Riccati equation to become computationally more efficient. Second, we show that for one class of problems explicit analytical solutions for the dynamic (and algebraic) Riccati equations can be obtained, and that for this class, subject to a rank condition, the optimal controls are a linear function of a scalar Riccati kernel.

The paper derives a simple rank expression that places an upper bound on the effective

[1] With the exception of Binder and Pesaran (2000) these papers focus on infinite horizon problems. The points raised in our paper apply to finite horizon problems as in Binder and Pesaran but are equally relevant for infinite horizon models.

dimensionality of the system for analytical and computational purposes: Prior computation of the rank of a composite matrix constructed from all coefficient matrices in the problem statement allows the researcher to establish this bound. The advantage is that one may readily determine up front whether the system has a simple analytical solution, or to what extent reformulation of the problem along the lines delineated here may reduce computation time or improve the transparency of the model.

Duality between Linear quadratic control (LQC) and Kalman filtering provides intuition for why the dimensionality of a particular system may be reduced. Consider the following class of Kalman problems.[2] An observation depends linearly on two unobserved state variables following stochastic processes: $w_t = y_{1t} + y_{2t}$. One may describe the uncertainty of the state by considering the conditional variances $\sigma_{1t}^2$, $\sigma_{2t}^2$ of the state variables and their conditional covariance $\sigma_{12t} = \sigma_{21t}$ (three numbers, stored in a *2 x 2* covariance matrix). However, conditional on having observed $w_t$, it is easy to derive from $y_{1t} = w_t - y_{2t}$ that $\sigma_{1t}^2 = \sigma_{2t}^2 = -\sigma_{12t}$ so that one number is sufficient to describe the state uncertainty. While the intuition for simplification here is straightforward, our rank expression implies a potentially complex interaction between the different singularities in the system that is not always intuitive.

The Kalman application also provides a class of problems for which our reduction approach may provide major computational advantages. Namely, for those models in which a Kalman filtering problem is embedded in a larger dynamic model that is not linear quadratic, such as an active learning model. Typically, such a model is analyzed with numerical dynamic programming methods in which the "curse of dimensionality" is a major impediment. Reducing

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[2] For an example see Claar's (2000) model of cyclical and natural unemployment rates.

the number of state variables in the grid then provides a substantial computational advantage.[3]

A reduction as proposed here was employed by Balvers and Cosimano (1994) in lowering the dimensionality of their active learning model, but the approach has not been systematically investigated. Mitchell (2000) derived explicit analytical solutions to the *2 x 1* linear-quadratic control problem (two target variables and one uncosted control in the control case, or two state variables and one identity in the Kalman case), but his results were not obviously generalizable. In this paper we significantly extend the class of LQC models which can be simplified or even solved analytically.

The paper is organized as follows. Section 2 derives the theorems that state how the dimensionality of the model can be reduced and by how much, and how under a certain rank condition the optimal feedback control matrix is linearly related to the reduced Riccati matrix. The analytical solution is given for one class of cases and examined in section 3. In Section 4 we conclude the paper by summarizing the advantages of our approach and discussing an algorithm of a MATLAB program (which is further described in Appendix C) that automates our technique for practical use.

## 2. Reduction of Dimension

### *2.1 The control problem*

In this section we show how to reduce the dimension of the Riccati equation of optimal control. In so doing we illuminate the underlying structure of the dynamics. Two initial lemmas establish the structure of the Riccati matrices, and Theorem 1 gives the reduced dynamics. The reduced problem is shown in Theorem 2 to be sometimes amenable to further simplification of

[3] The relevant state variables in active learning models include the conditional covariances and (sometimes) the conditional means of the relevant underlying state variables. Reduction of the Riccati dimension does not reduce the number of conditional means but significantly reduces the number of covariances to be considered. As an example, in Wieland (2000) the state space discussed in his Appendix A3 could be reduced from four to two state variables if there were no measurement error in his equation (1).

the solution for the control feedback matrix. Theorem 1* deals with a further reduction of the Riccati matrix dimension which is possible under some conditions.

The reduction that we present is separate from the concept of reducing a system to "minimal" form for optimal control or Kalman filtering. A system is in minimal form if the number of state variables describing the system cannot be reduced any further. This form is attained if and only if each state variable is controllable (meaning loosely that the control variables can directly or indirectly impact each state variable) as well as observable (meaning loosely that each state variable is relevant in affecting the objective). See Hannan and Deistler (1988). Our Riccati reduction, however, applies even if the system is controllable and observable and, hence, minimal. This reduction can be achieved because the effective dimension of the Riccati matrix (the dimension of the Riccati "kernel") is less than the dimension of the state vector even if the latter is minimal.

### *2.2 The LQC problem*

We start with a general finite horizon, stochastic LQC problem, Problem 1. The square coefficient matrices $\bar{K}$, $R$, and $\bar{A}$ are allowed to be singular (but matrices $\bar{K}$ and $R$ must be symmetric and satisfy the second-order condition in Appendix A), and $J$, $\bar{C}$, and $\bar{G}$ need not have full row or column rank. Equivalent restrictions apply for the time $T$ terminal coefficient matrices. Problem 1 is:

$$\text{(1a)} \qquad V(\bar{y}_s, s) = \underset{\{\bar{u}_t\}_{s+1}^{T}}{Min} \; E_s \left[ \left( \sum_{t=s+1}^{T-1} \beta^t \left[ \tfrac{1}{2}(\bar{y}_t^{\prime} \bar{K} \bar{y}_t) + (\bar{y}_t^{\prime} J \bar{u}_t) + \tfrac{1}{2}(\bar{u}_t^{\prime} R \bar{u}_t) \right] \right) + \tfrac{1}{2} \beta^T (\bar{y}_T^{\prime} \bar{K}_T \bar{y}_T + 2 \bar{y}_T^{\prime} J_T \bar{u}_T + \bar{u}_T^{\prime} R_T \bar{u}_T) \right],$$

$$\text{(1b)} \qquad \textit{subject to} \quad \bar{y}_t = \bar{A} \bar{y}_{t-1} + \bar{C} \bar{u}_t + \bar{G} \varepsilon_t, \quad t = s+1, \ldots, T, \quad \bar{y}_s,$$

where the state vector $\bar{y}_t$ is $\bar{n}$ x 1, the control vector $\bar{u}_t$ is $k$ x 1, and the vector of i.i.d. random shocks is $g$ x 1; the coefficient matrices are conformable. Appendix A provides the a standard transformation of the above LQC problem into a decision problem which has no control costs or discounting and no stochastic terms.

The simple transformation in Appendix A translates LQC Problem 1 into Problem 2 below, which has a state vector containing all costed variables. The transformation will have increased the size of the state vector but additional standard transformations that reduce the size of the state vector can be applied to render the problem observable and controllable so that the dimension of the state vector is "minimal." This process (which we have programmed in MATLAB) initially may or may not have decreased the dimension of the state vector. However, the idea is that starting from a dimension $n$ of the state vector, the dimension of the Riccati matrix in LQC Problem 2 can be reduced to a size of $n$-$k$ where $k$ is the size of the control vector. This dimension is typically lower than the dimension of the "minimal" state vector.

If the control cost matrix, the state cost matrix, and the transition matrix are all of full rank and the problem is minimal, then the addition of the control variables plus the transformations to again make the problem observable and controllable must add the size $k$ of the control vector to the state vector; and in the current section we reduce the Riccati dimension by $k$ so that the net gain is zero. However, if singularities exist in the control cost matrix, the state cost matrix, or the transition matrix, then our technique usually reduces the dimension of the Riccati dynamics, sometimes to the point where an analytical solution is possible. In the large class of LQC problems where the initial problem has no control costs and an invertible state cost matrix, Problem 2 below applies directly and our following reduction technique reduces the dimension of the Riccati matrix from $n$ to $n$-$k$ or less.

LQC Problem 2 is:

(2a) $$V(y_s, s) = \underset{\{u_t\}_{s+1}^{T}}{Min} \left( ½(y_T^{/} K_T y_T) + \sum_{t=s+1}^{T-1} [½(y_t^{/} K y_t)] \right) ,$$

(2b) $$\textit{subject to} \quad y_t = A y_{t-1} + C u_t , \quad t = s+1 , ..., T , \quad y_s \ \textit{given},$$

where the state cost matrices $K$ and $K_T$ and the transition matrix $A$ are $n \; x \; n$ , the control multiplier matrix $C$ is $n \; x \; k$, the state vector $y_t$ is $n \; x \; 1$ , and the control vector $u_t$ is $k \; x \; 1$. The cost matrix $K$ and the terminal state cost matrix $K_T$ are positive definite and hence nonsingular, and $C$ has full column rank; $A$ need not have full rank.[4]

It is well known [for example, Chow (1975)] that the optimal controls for Problem 2 are given by:

(3) $$u_t^{opt} = -(C^{/} H_t C)^{-1} C^{/} H_t A y_{t-1} \equiv -F_t y_{t-1} , \qquad t \leq T ,$$

(4) $$H_{t-1} = K + A^{/} H_t A - A^{/} H_t C (C^{/} H_t C)^{-1} C^{/} H_t A, \quad H_T = K_T, \quad t \leq T,$$

where the symmetric $n \; x \; n$ matrix $H_t$ is positive definite.

### *2.3 Some initial intuition*

Combining the equations of motion (2b) with the optimal control choice (3) yields the optimally controlled state variables:

(5) $$y_t^{opt} = [I - C(C^{/} H_t C)^{-1} C^{/} H_t ] A y_{t-1} .$$

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[4] We make the assumption that *K* is invertible in the text, even though this is not necessary to apply our MATLAB program in which we employ what is effectively a generalized inverse for *K*. To avoid additional complexity we do not use generalized inverses in the paper. For practice purposes, the process of rendering the problem minimal often produces an invertible *K* matrix. The assumption that *C* has full column rank is required for the necessary second-order condition that *C'KC* be positive definite. If *C* is not of full column rank, second-order conditions must fail because of redundant controls.

Pre-multiplying both sides of equation (5) by $C'H_t$ gives that

$$(6) \qquad C'H_t y_t^{opt} = 0_{k,n}.$$

This means that the optimal control choices in each period generate $k$ linear dependencies among the $n$ state variables (remember that $C$ has full column rank $k$ and that $H_t$ has full rank $n$): in LQC without control costs, associated with each control is a linear dependency; thus, the essential dynamics of the decision problem has dimension $n$ - $k$.

We apply our technique to a general formulation that may not be minimal. Whereas we might start from a less general, minimal formulation, this restriction would needlessly limit the applicability of the technique because minimality of the problem does not affect the analysis that follows. Our MATLAB program, however, can convert any LQC problem into minimal form and to understand more clearly the parsimony of our approach one may think of minimality as being imposed.

How, then, is it possible that a dimension less than the minimal number of state variables is needed to describe the Riccati dynamics of the model? The answer, with details provided in the following, is that only $n$ - $k$ state variables are involved in obtaining the value function and capturing the dynamics of what we call the Riccati kernel. The remaining $k$ state variables are necessary only to capture the state dynamics (as implied by minimality).

### *2.4 The basic reduction*

Equation (4) can be written as equations (7) and (8):

$$(7) \qquad H_{t-1} = K + A'P_t A , \qquad t \leq T ,$$

$$(8) \qquad P_t = H_t - H_t C (C'H_t C)^{-1} C'H_t , \qquad t \leq T.$$

The approach in this paper is to exploit restrictions inherent in the $P_t$ matrix to simplify the solution of Problem 2.[5] By equation (8) we have:

$$(9) \qquad P_t C \equiv \begin{pmatrix} P_{1t} & P_{2t} \\ P_{3t} & P_{4t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0_{n,k} , \qquad t \le T,$$

where, defining $q = n - k$, we have that $P_{1t}$ is $q \times q$, $P_{2t}$ is $q \times k$, $P_{3t} = P_{2t}^{/}$, and $P_{4t}$ is $k \times k$; $C_1$ is $q \times k$, and $C_2$ is $k \times k$. $P_{1t}$ and $P_{4t}$ are symmetric.

Since $C$ is of full column rank, there is at least one $k \times k$ sub-matrix of $C$ that is invertible. Proper prior arrangement of the $y_t$ vector (and concomitant arrangement of $C$, $A$, $K$, and $K_T$) puts these $k$ rows together at the bottom of $C$ guaranteeing that $C_2$ is invertible. We then derive:

**LEMMA 1 (REDUCTION TO THE DYNAMIC CORE OF $P_t$).** The $n \times n$ matrix $P_t$ can be written as:

$$(10) \qquad P_t = M \Phi_t M^{/} , \quad \Phi_t \equiv P_{1t} , \qquad t \le T$$

$$(11) \qquad M \equiv \begin{bmatrix} I_{n-k} \\ -(C_2^{/})^{-1} C_1^{/} \end{bmatrix},$$

where $M$ is an $n \times q$ matrix, and $\Phi_t$ is invertible with dimensions $q \times q$.

*Proof*. See Appendix B1. ■

[5] The symmetric $n \times n$ $P_t$ matrix is not typically employed in dealing with optimal control problems, but in the dual Kalman filtering context has the familiar interpretation of the covariance matrix for the unobserved state variable for the current period conditional on current information (while $H_t$ is the covariance matrix conditional on the previous period's information).

It is important to relate the Riccati kernel $\Phi_t$ to the solution of LQC Problems 1 and 2 – equations (3) and (4) – in a meaningful way. Lemma 2 provides a useful link.

**LEMMA 2 (RELATING $\Phi_t$ AND $H_t$).** The $q \times q$ matrix $\Phi_t$ in equation (10) is positive definite and is given by:

$$(12) \qquad \Phi_t = (M' H_t^{-1} M)^{-1}, \qquad t \leq T.$$

*Proof.* See Appendix B2. ■

Employing Lemmas 1 and 2 we provide the dynamics of $\Phi_t$.

**THEOREM 1 (DYNAMICS OF $\Phi_t^{-1}$).** For all $t \in \{s+1, T\}$ we have :

$$(13) \qquad \Phi_{t-1}^{-1} = B_1 - B_2'(\Phi_t^{-1} + B_3)^{-1} B_2, \quad \Phi_T^{-1} = M' K_T^{-1} M,$$

with $B_1 = M' K^{-1} M$, $B_2 = M' A K^{-1} M$, $B_3 = M' A K^{-1} A' M$, and $M$ given by equation (11).

*Proof.* See Appendix B3. ■

The reduced Riccati equation (13) has dimension smaller than that of the original Riccati equation (4).

### *2.6 The case of nonsingular $B_2$*

The $B_i$ matrices in Theorem 1 are all $q \times q$ and only $B_2$ is not symmetric. $B_1$ is positive definite and $B_3$ is positive semi-definite. By Sylvester's inequality (equation B1.1),

$B_2$ (= $M' A K^{-1} M$) can be of full or less than full rank regardless of whether $A$ has full rank. (However, if $rank\,(A) < q$ then $B_2$ is certainly singular. Section 3.1 below provides an example.)

The sequence of reduced Riccati matrices obtained in Theorem 1 can be used with equations (3), (7), and (10) to obtain the sequence $\{u_t^{opt}\}$ of optimal controls. However, given the transformations employed here there is a more convenient way of calculating the optimal controls when $B_2$ has full rank (=$q$), as in this case the feedback matrix can be shown to be linear in $\Phi_t$:

**THEOREM 2 (LINEAR CALCULATION OF FEEDBACK MATRIX).** If $rank\,(B_2) = q$, then in the optimal control solution $u_t^{opt} = -F_t\, y_{t-1}$, the feedback matrix $F_t$ is linear in $\Phi_t$ for all $t \leq T$ - 1 :

$$F_t = -W M \Phi_t M' A + W K A, \qquad t \leq T-1, \tag{14}$$

with $W = (C'C)^{-1} C' K^{-1} [I - A' M (M' K^{-1} A' M)^{-1} M' K^{-1}]$.

*Proof.* See Appendix B4. ■

Thus computation of the sequence $\{F_t\}$ of control feedback matrices involves first computing $F_T = (C' K_T C)^{-1} C' K_T A$ from equation (3) with $H_T = K_T$, next iterating equation (13) to get $\{\Phi_t\}$, and then using equation (14) to obtain the remainder of the feedback matrix sequence. The value of Theorem 2 is that it allows the feedback matrix sequence to be calculated linearly and that it may facilitate comparative statics analysis.

*2.7 Further reduction when $B_2$ is singular*

We now consider the case in which $B_2$ in equation (13) is singular. This provides the opportunity for further reduction of the size of the Riccati matrix: the *n x n* Riccati matrix *H* has already been reduced to the *q x q* Riccati matrix $\Phi$; if the *q x q* matrix $B_2$ has rank $r < q$, we can further reduce $\Phi$ to an *r x r* matrix to be denoted $\Phi^*$.

First put the *q x q* matrix $B_2$ in standard form:

$$I_r^q = Q B_2 S \ , \quad \text{where } I_r^q \equiv \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \ , \quad S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} .$$

Here *Q* and *S* are invertible *q x q* matrices and $S_1$ is *r x r*. *S* must be arranged such that $S_4$ is invertible (which may require a row and column rearrangement as discussed in the footnote of Appendix B5). Then we have:

**THEOREM 1* (FURTHER REDUCTION OF $\Phi_t$).** If $B_2$ has less than full rank, the dimension of the time-varying kernel of the Riccati equation is no higher than that of $\Phi_t^*$, which is *r x r* where $r = \text{rank}\,(B_2)$. $\Phi_t^*$ is positive definite, with $\Phi_t^{*\,-1} \equiv Z'\Phi_t^{-1}Z$ where *Z* is defined below, and its dynamics is described by:

$$(15) \qquad \Phi_{t-1}^{*\,-1} = B_1^* - B_2^{*\prime}(\Phi_t^{*\,-1} + B_3^*)^{-1} B_2^* \ , \qquad t \leq T - 1,$$

$$\Phi_{T-1}^{*\,-1} = Z'[B_1 - B_2'(\Phi_T^{-1} + B_3)^{-1} B_2] Z;$$

$$\text{with} \quad B_1^* = Z'\{B_1 - B_2'(B_1+B_3)^{-1}B_2 + B_2'(B_1+B_3)^{-1} \cdot$$

$$M^*[M^{*'}(B_1+B_3)^{-1}M^*]^{-1}M^{*'}(B_1+B_3)^{-1}B_2\}\,Z,$$

$$B_2^* = [M^{*'}(B_1+B_3)^{-1}M^*]^{-1}M^{*'}(B_1+B_3)^{-1}B_2\,Z\,,$$

$$B_3^* = [M^{*'}(B_1+B_3)^{-1}M^*]^{-1} - Z'B_1Z,$$

$$\text{where } Z = \begin{pmatrix} I_r \\ 0_{q-r,r} \end{pmatrix} \text{ and } M^* = \begin{pmatrix} I_r \\ -S_4^{-1'}S_2' \end{pmatrix} \quad (\text{so } M^* \text{ is } q \times r).$$

*Proof*. In Appendix B5. ■

In equation (15), $B_2^*$ may or may not be invertible. (Appendix B5 gives an example in which it is not.) In either case, Theorem 1* may be combined with equations (3), (7), (10), and (B5.3) to obtain the sequence of optimal controls based on $\{\Phi_t^*\}$. If $B_2^*$ is not invertible, we can further reduce the dimensionality of the problem by repeatedly applying the reduction process of Theorem 1* until the reduced analog of $B_2^*$ either has zero rank or is invertible. Therefore the effective dimension of the original problem is less than or equal to the rank of $B_2$.

Note that the proof of Theorem 1* required that the initial condition be stated as of period *T - 1*. Thus in iterating $H_t$, $H_T$ equals $K_T$, while for *t # T - 1* $\Phi_t^*$ is iterated and then used to compute $\Phi_t$, which in turn is used to compute $H_t$. Any further reduction, shifts back one more period the time of the initial condition for reduced Riccati iteration.

There is no counterpart to Theorem 2 for the case in which any reduction beyond that of Theorem 1 has been applied, because the proof of Theorem 2 rests crucially on invertibility of $B_2$ and cannot be generalized to rely on invertibility of $B_2^*$.

This section has shown how to reduce the size of the Riccati matrix of optimal control, thereby simplifying computation of the Riccati iteration and solution and revealing the underlying structure of the dynamics. To obtain these results we did not require invertibility of the matrix of control costs or of the transition matrix.

## 3. IMPLICATIONS

### *3.1 Effective dimension of the system*

By Theorem 1*, the upper bound on the effective dimension of the system (the size of $\Phi_t^*$ or of $\Phi_t$ if $B_2$ has full rank) is given by the rank of $B_2 \equiv M'AK^{-1}M$ with $M$ defined in equation (11). This bound may be determined in advance – that is, before theoretical appraisal, estimation, numerical analysis, or explicit solution of the model. A general indication of the rank of $B_2$ is obtained by repeated application of Sylvester's inequality (see equation B1.1) to the definition of $B_2$ given in Theorem 1. Recalling that $n$ represents the dimension of the state vector and $k$ the number of controls, Sylvester's inequality yields:

$$(18) \qquad rank(A) - 2k \le rank(B_2) \le \min[n-k,\ rank(A)].$$

Scalar Riccati dynamics is guaranteed if $n - k$ (the size of $B_2$) = $1$ (or, of course, if $rank(A) = 1$). This case will be discussed in the next sub-section.

Before we discuss the scalar case, we present a simple example to illustrate the bounds implied by equation (18). Consider a case with $n = 3$, $k = 1$, $C' = (0 \;\; 0 \;\; 1)$, and $K = I_3$. The *3 x 3* matrix *A* is unrestricted. Note that it is always possible, starting from any like-sized problem with any *C*, to transform the control and state vectors so that $C' = (0 \;\; 0 \;\; 1)$. Then, $B_2 = A_1$, where $A_1$ is the *2 x 2* upper left block of *A*. Consequently, there is an infinitude of *A* matrices for which any of the following hold: (a) $rank\,(B_2) = rank\,(A_1) = n - k = 2$, when the two *1 x 2* rows of $A_1$ are independent, with $rank\,(A)$ equaling either three or two; (b) $rank\,(B_2) = rank\,(A_1) = rank\,(A) - 2\,k = 1$, when $A$ has rank one, two, or three and the two *1 x 2* rows of $A_1$ are dependent; and (c) $rank\,(B_2) = rank\,(A_1) = rank\,(A) - 2\,k = 0$, when all four elements of $A_1$ are zero so that $A$ must be singular with either rank one or two.

### *3.2 Analytical solution when rank ($B_2$) # 1*

When $B_2 = M' A K^{-1} M$ has rank equal to or less than one, the LQC problem allows scalar-based analytical solution. When the rank of $B_2$ is equal to *zero* because $B_2 = 0$, Theorem 1 directly shows that $\Phi_t$ does not evolve. When the rank of $B_2$ is equal to *one*, Theorem 1 applies if $n - k = 1$ (so $B_2$ has full rank) and Theorem 1* applies if $n - k > 1$ (so $B_2$ has less than full rank). In what follows we discuss the case $n - k = 1$, but if the rank of $B_2$ is less than full and equals 1, Theorem 1* applies and the results below all continue to hold if we replace the $B_i$ by $B_i^*$.

**THEOREM 3 (ANALYTIC SOLUTION WHEN RANK ($B_2$)= 1).** If *rank* $(B_2) = 1$ and $n - k = 1$, the solution for $\Phi_t$ is:

$$(19) \qquad \Phi_{t-1} = [1 + B_3 \Phi_t] / [B_1 + (B_1 B_3 - B_2^2) \Phi_t], \qquad t \le T,$$

where Theorem 1 defines $B_1$, $B_2$, and $B_3$, which are scalar in this case. If *rank* $(B_2) = 1$ and $n - k > 1$, the solutions are given by replacing $B_1$, $B_2$, and $B_3$ in equation (19) by the scalars $B_1^*$, $B_2^*$, and $B_3^*$ defined in Theorem 1*.

*Proof*. Equation (13) implies that $\Phi_t$ in equation (19) is scalar. ■

Mitchell (2000) finds the solution to a scalar equation of the form of equation (19) as follows. Consider first the case of $B_1 B_3 - B_2^2 \neq 0$, so that $\Phi_t$ evolves nonlinearly (unless $B_2 = 0$). Let $x_t = 1/(c + \Phi_t)$ and hence $\Phi_t = (1 - c x_t)/x_t$, where $c = (B_1 - B_3 + r)/[2(B_1 B_3 - B_2^2)]$ and $r = [(B_1 - B_3)^2 + 4(B_1 B_3 - B_2^2)]^{1/2}$. Then use $\Phi_t = (1 - c x_t)/x_t$ on both sides of equation (19) to obtain a linear equation of evolution for $x_t$:

$$(20) \qquad x_{t-1} = \frac{2(B_1 B_3 - B_2^2)}{B_1 + B_3 + r} + \left( \frac{B_1 + B_3 - r}{B_1 + B_3 + r} \right) x_t, \qquad t \le T,$$

with solution

$$(21) \qquad x_t = \frac{B_1 B_3 - B_2^2}{r} + \left( x_T - \frac{B_1 B_3 - B_2^2}{r} \right) \left( \frac{B_1 + B_3 - r}{B_1 + B_3 + r} \right)^{T-t}, \qquad t \le T.$$

Then the solution for $\Phi_t$ is found by putting equation (21) into $\Phi_t = (1 - cx_t)/x_t$.

It is also possible for equation (19) to give *linear* evolution of $\Phi_t$. This occurs if and only if $B_1 B_3 - B_2^2 = 0$. In this linear case the solution of equation (19) for $\Phi_t$ is obvious and the eigenvalue is $B_3/B_1$, which [as Mitchell (2000) shows] may or may not be less than one in magnitude so the linear case may or may not be stabilizable.

To examine the nature of the scalar dynamics, first derive from equation (19):

$$(22) \qquad d\Phi_{t-1}/\, d\,\Phi_t \;=\; B_2^2 \,/\, [B_1 \,+\, (B_1 B_3 - B_2^2)\,\Phi_t\,]^2 \;\geq\; 0 \qquad .$$

Equation (22) allows us to identify three qualitatively distinct cases:

*Case 1*: $B_2 = 0$. This case is covered equally well by Theorem 1 or Theorem 1*. Equation (19) collapses to $\Phi_{t-1} = 1/B_1$ which is constant. Figure 1(a) shows the dynamics of $\Phi_t$: the steady state is reached in one iteration.

*Case 2*: $B_2 \neq 0$ and $B_1 B_3 - B_2^2 \neq 0$. Note that $B_1 B_3 - B_2^2$ cannot be negative: we know

$$B_3 - B_2 B_1^{-1} B_2' \;=\; M'AK^{-1}[K - M(M'K^{-1}M)^{-1}M']K^{-1}A'M \;=\; M'AC(C'KC)^{-1}C'A'M,$$

where the last equality follows from substituting equation (12) into equation (10) and the result into equation (8), evaluating the resulting identity at $H_t = K$, subtracting $K$ from both sides, and pre- and post-multiplying both sides by $K^{-1}$. Hence, $B_3 - B_2 B_1^{-1} B_2'$ is positive semi-definite, and so in this scalar case multiplying this expression by the positive scalar $B_1$ establishes $B_1 B_3 - B_2^2 \geq 0$. Then in this case 2 equation (22) implies that $d\Phi_{t-1}/\, d\,\Phi_t > 0$ and

$d^2\Phi_{t-1}/\,d\,\Phi_t^2\;<\;0$; and as $\Phi_t\;\to\infty$ we have $d\Phi_{t-1}/\,d\,\Phi_t\,\to\,0$. Thus, the time path is monotonic and convergent as displayed in Figure 1(b).

*Case 3*: $B_2\;\neq\;0$ and $B_1B_3\,-\,B_2^2\;=\;0$. Now by equation (19), $\Phi_{t-1}\;=\;(1/B_1)\;+\;(B_3/B_1)\,\Phi_t$ so evolution is linear. This permits the stable case of $B_3 < B_1$ shown in Figure 1(c) (noting that both $B_1\,(=\,M'K^{-1}M)$, and $B_3\,(=\,M'AK^{-1}A'M)$ must be nonnegative given positive definite $K$) as well as the unstable case of $B_3\;\geq\;B_1$, also shown in Figure 1(c).

We have shown here how to solve the case of *n - k = 1* analytically, which was heretofore done only for the *n = 2, k = 1* case by Mitchell (2000). In addition, we have shown how, due to potential singularities in the transition matrix and its interactions with the cost matrix, other apparently more complex problems can also be solved analytically if the effective dimensionality equals *1*.

### *3.3 An example*

To illustrate some of the advantages of our reduction technique, consider a simple extended IS/LM model with nominal wage rigidities (see DeLong and Summers, 1986, for a similar model):

(23) $\quad m_t - p_t\;=\;q_t - a\,i_t$ (LM)

(24) $\quad q_t\;=\;b\;-\;cr_t$ (IS)

(25) $\quad q_t\;=\;s_t\;+\;dp_t$ (Aggregate Supply)

(26) $\quad s_t\;=\;\rho\,s_{t-1}\;+\;\varepsilon_t$ (Supply Shocks)

(27) $\quad i_t\;=\;r_t\;+\;(E_tp_{t+1} - p_t)$ (Nominal Interest Rate).

The government determines the money supply at each time to minimize the discounted value of deviations in output from target and deviations in inflation from zero:

$$(28) \qquad V(s) = \underset{\{m_t\}_{s+1}^{\infty}}{Min} \left( ½ \; E_s \sum_{t=s+1}^{\infty} \beta^t [(q_t - q)^2 + h(p_t - p_{t-1})^2] \right).$$

All variables have their standard definitions and are in log terms, and parameters are positive. The random variables have a mean of zero and are i.i.d. To convert the model to the desired format of Problem 2 consider that rational expectations implies

$$(29) \qquad E_t p_{t+1} = p_{t+1} - \eta_{t+1}.$$

Clearly $\eta_t$ is (perfectly) correlated with $\varepsilon_t$ but this does not concern us here. Simplifying yields the following equations of motion:

$$(30) \quad p_{t+1} = -(b/c) + [(1/a) + (1/c)]q_t + [1 + (1/a)]p_t - (1/a)m_t + \eta_{t+1},$$

$$(31) \quad q_{t+1} = -(db/c) + [\rho + (d/a) + (d/c)]q_t + d[1 - \rho + (1/a)]p_t - (d/a)m_t + \varepsilon_{t+1} + d\eta_{t+1}.$$

The government objective suggests that we add a constant and a lagged price level variable as state variables. Hence:

$$(32) \qquad \bar{y}_t = \begin{pmatrix} 1 \\ p_t \\ p_{t-1} \\ q_t \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(b/c) & 1 + (1/a) & 0 & (1/c) + (1/a) \\ 0 & 1 & 0 & 0 \\ -(d\,b/c) & d(1-\rho) + (d/a) & 0 & \rho + (d/c) + (d/a) \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} 0 \\ -(1/a) \\ 0 \\ -(d/a) \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} 0 \\ \eta_t \\ 0 \\ \varepsilon_t + d\,\eta_t \end{pmatrix}, \quad \bar{K} = \begin{pmatrix} q^2 & 0 & 0 & -q \\ 0 & h & -h & 0 \\ 0 & -h & h & 0 \\ -q & 0 & 0 & 1 \end{pmatrix}, \quad \bar{u}_t = m_{t-1}.$$

For simplicity we set the discount factor β equal to 1.

The problem is not in minimal form since two of the four states are not controllable as can be checked by inspecting the rank of the controllability matrix $[\bar{C}\,|\,\bar{A}\,\bar{C}\,|\,\bar{A}^2\,\bar{C}\,|\,\bar{A}^3\,\bar{C}]$ which is two. Based on Rubio (1971, 194-206), transforming the decision problem to minimal form requires finding two independent columns from the controllability matrix and calling this *4 x 2* matrix $S_c$. Then find a pseudo inverse $V_c$ (*2 x 4*) so that $V_c\,S_c = I$. The system obtained as: $K = S_c'\,\bar{K}\,S_c$, $C = V_c\,\bar{C}$, $A = V_c\,\bar{A}\,S_c$, $y_t = V_c\,\bar{y}_t$, generates the same optimal controls and loss function as the original problem but based on fewer state variables.

We find $S_c = [\bar{C}\,|\,\bar{A}\,\bar{C}] = \begin{pmatrix} 0 & 0 \\ 1-\alpha & (1-\alpha)(\alpha+\gamma\,\delta) \\ 0 & 1-\alpha \\ \delta(1-\alpha) & \delta(1-\alpha)(\alpha+\gamma\,\delta) \end{pmatrix}$, and easily choose some $V_c$ matrix, we pick $V_c = \begin{pmatrix} 0 & 1/(1-\alpha) & (\alpha+\gamma\,\delta)/(1-\alpha) & 0 \\ 0 & 0 & 1/(1-\alpha) & 0 \end{pmatrix}$, such that $V_c\,S_c = I$,

where we have defined $\alpha = 1 + (1/a)$, $\gamma = (1/c) + (1/a)$, $\delta = d$.

This yields

$$K = \begin{pmatrix} (1-\alpha)^2\,(\delta^2+h) & (1-\alpha)^2[\,(\delta^2+h)(\alpha+\gamma\,\delta) - h\,] \\ (1-\alpha)^2[\,(\delta^2+h)(\alpha+\gamma\,\delta) - h\,] & (1-\alpha)^2[(\alpha+\gamma\,\delta)^2(\delta^2+h) + h - 2h(\alpha+\gamma\,\delta)] \end{pmatrix}, \tag{33}$$

$$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;,\;\; A = \begin{pmatrix} 0 & 0 \\ 1 & \alpha + \gamma\delta \end{pmatrix},\;\; y_t = \begin{pmatrix} 1/(1-\alpha) & (\alpha + \gamma\delta)/(1-\alpha) \\ 0 & 1/(1-\alpha) \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} .$$

Two state variables can be dropped. In this case the constant and $q_t$, which from equation (25) is tied directly to $p_t$. In principle, judicious choice of state variables should accomplish a minimal formulation without requiring a transformation, but this is not always easy. The problem now is clearly minimal since *K* is full rank implying observability and $[C \,|\, A\,C] = I$ so clearly full rank implying controllability.

Since there is one control variable, our basic reduction implies that the dimension of the Riccati kernel governing the dynamics can be reduced from *n* to *n-k*, that is, from two to one, so that explicit solution is possible. Applying Theorem 1, we first obtain $M' = (0 \;\; 1)$. Then we can find

$$(34) \qquad B_1 = \frac{1 + (\delta^2/h)}{(1-\alpha)^2\,\delta^2} > 0\,, \quad B_2 = \frac{1}{(1-\alpha)^2\,\delta^2} > 0, \quad B_3 = \frac{1}{(1-\alpha)^2\,\delta^2} > 0\,.$$

Further, $B_1\,B_3 - B_2^2 = \dfrac{1}{(1-\alpha)^4\,\delta^2\,h} > 0$. Hence, we know that Case 2 applies as depicted in Figure 1(b). In addition, since $B_2$ is full rank, we can apply Theorem 2:

$$(35) \qquad F_t = \begin{pmatrix} 1 \\ \alpha + \gamma\delta \end{pmatrix}' [\,(\alpha - 1 + \gamma\,\delta) + \Phi_t\,/h(1-\alpha)^2\,],$$

The feedback control policy for both of the state variables is linear in the (scalar) Riccati kernel. The Riccati kernel can be obtained from equation (19) as:

$$(36) \qquad \Phi_{t-1} = (1-\alpha)^2 h\,[(1-\alpha)^2 \delta^2 + \Phi_t] / [(1-\alpha)^2(\delta^2 + h) + \Phi_t].$$

Since the nonstabilizable case of Figure 1(c) does not arise in this model, we could solve for the reciprocal of the steady state value $\Phi$ of the Riccati kernel:

$$(37) \qquad \Phi^{-1} = \{1 + [1 + (4h/\delta^2)]^{1/2}\} / [2(1-\alpha)^2 h].$$

Note that the full algebraic Riccati matrix is not needed. It can be obtained from equation (37) via equations (7) and (10) but is quite complicated.

This example has demonstrated the use of our reduction technique in the context of a relatively simple macroeconomic model which can be expressed in terms of four state variables and one control variable, and has shown how to reduce the Riccati matrix to its scalar kernel and to express the control feedback matrix linearly in terms of the Riccati kernel. Additional examples (including two based on Amman and Neudecker, 1997, and Ljungqvist and Sargent, 2000) and a MATLAB program (see Appendix C) are available from the authors at http://www.be.wvu.edu/divecon/econ/balvers/riccatimatlab.htm .

## 4. Summary and Conclusion

A procedure has been presented for simplifying and solving LQC models. The procedure is automated in MATLAB and can be summarized in the following algorithm:

*Step 1*. If necessary, transform the LQC problem to fit the structure of equations (2). Further transformations can, but need not be, employed to make the problem minimal.

*Step 2*. First obtain $M$ from equation (11) and subsequently obtain $B_1$, $B_2$, $B_3$, and $\Phi_T^{-1}$ as given in Theorem 1.

*Step 3*. If $B_2$ has full rank find $\{\Phi_t\}$ from Theorem 1. Then skip to *Step 8*.

*Step 4*. If $B_2$ does not have full rank, find the *S* matrix by transforming $B_2$ into standard form and extract $S_4^{-1}$ and $S_2$. (If $S_4^{-1}$ does not exist consider footnote 4.)

*Step 5*. Obtain $M^*$, $B_1^*$, $B_2^*$, $B_3^*$, and $\Phi_{T-1}^{*-1}$ as given in Theorem 1*. If $B_2^*$ is not invertible repeat *Steps 4* and *5*.

*Step 6*. Find $\{\Phi_t^{*-1}\}$ from Theorem 1*.

*Step 7*. Employ equation (B5.3) to deduce $\{\Phi_t\}$ from $\{\Phi_t^{*-1}\}$.

*Step 8.* Substitute $\{\Phi_t\}$ into equation (10) to find $\{P_t\}$ and then use equation (7) to generate $\{H_t\}$, if $\{P_t\}$ and $\{H_t\}$ are needed. The end matrix $H_T$ is generated as $H_T = K_T$, and $H_{T-1}$ is generated from $\Phi_T$ via equations (7) and (10).

*Step 9.* If $B_2$ has full rank, find the feedback matrix sequence $\{F_t\}$ from equation (3), or, for $t \leq T-1$, from Theorem 2. The optimal control vector $u_t^{opt}$ equals $-F_t y_{t-1}$.

*Step 9'.* If $B_2$ is singular, use $\{H_t\}$ from *Step 8* and use equation (3) to find the $\{F_t\}$ matrix sequence.

This procedure provides a simple calculation (the rank of $B_2$) to establish an upper bound on the effective dimension of the problem. It is then possible to find in advance, without computing the solution, how complicated or simple the dynamics and steady state equations are. It is applicable even when the transition matrix, $\bar{A}$, the control cost matrix, $R$, or the state cost matrix, $\bar{K}$, are singular and in these cases usually reduces the dimensionality of the Riccati dynamics.

This reduction of the Riccati kernel has several computational and analytical advantages. First, in cases where an LQC problem is embedded in a larger dynamic programming model, such as arises in active learning problems, the curse of dimensionality implies that any reduction in the size of the Riccati kernel generates substantial computational savings. Second, current techniques for generating numerical solutions to the Riccati equation typically work faster when the dimension of the Riccati dynamics is lower (Note, however, that the matrices are more compact so the benefits of using any techniques that take advantage of sparsity are reduced).

Third, it is possible that numerical accuracy is increased because only elementary row and column operations are needed for solution and because most of the operations are imposed on lower-dimension matrices. Patel, Laub, and van Dooren (1994) point out the numerical advantage of working with smaller-order matrices but also emphasize the numerical instabilities that arise from roundoff errors. Accuracy depends accordingly on condition numbers and on a variety of other factors so is more easily judged on a case-by-case basis. It remains an issue for future research to determine for what class of economic problems the particular matrix inversion required at each iteration involves a sufficiently well conditioned matrix. Our approach calls for a few initial transformations, involving inverses of matrices that may or may not be ill-conditioned, but subsequently performs iterations on smaller-order matrices.

Fourth, analytical solutions can be obtained if the Riccati kernel is of dimension one (or zero). Such analytical solutions aid economic intuition. Fifth, our approach makes it easier to impose certain numerical restrictions on the coefficient matrices to construct special cases for which the Riccati dimension is one. These solvable cases provide computational advantages by allowing a check on the numerical accuracy of a particular solution algorithm. Sixth, the linearity of the optimal controls in the Riccati kernel may aid in theoretical comparative statics analysis and may be particularly efficient in policy-improvement solution algorithms (see for instance Ljungqvist and Sargent, 2000, p.56).

Finally, our approach can be applied even when any of the control cost matrix, the state cost matrix, or the transition matrix are singular. If the control cost matrix, the state cost matrix, and the transition matrices are invertible, the solution technique for LQC problems of Binder and Pesaran (2000) can be compared to ours in that it is applicable to finite horizon models and requires merely elementary row and column operations. Their approach, in terms of computationally demanding operations, requires only the inversion of five matrices (four $n \times n$ and one $k \times k$) to find the optimal control solution for the first period (for all periods if the problem is deterministic). Our approach is more demanding in that it involves inversion of five matrices (two $n \times n$ plus three $k \times k$) as well as $T - 1$ additional $n \times n$ inversions (one for each additional period) to obtain the optimal control for the first period. If optimal controls are calculated for all periods, however, our approach requires no additional inverses, but the Binder and Pesaran approach for a stochastic model involves $T - 1$ additional $n \times n$ inversions, making it about as demanding as ours. The benefits of our approach come into play when there are singularities in the control cost matrix, the state cost matrix, or the transition matrix. The Binder and Pesaran approach does not apply in these cases, whereas our approach becomes more efficient: it involves a fixed computational cost of inversion of five matrices (two $n \times n$ plus three $k \times k$) plus a variable cost of $T - 1$ additional $(n\text{-}k) \times (n\text{-}k)$ inversions. If $B_2$ is not invertible, the dimension of the variable-cost inversions decreases further, although one or two additional fixed-cost inversions (of dimension less than $n$) are needed.

## Appendix A: Transformation of LQC Problem 1 to LQC Problem 2

First, augment the state vector with the control vector and redefine variables to suppress the discount factor. We can then write Problem 1 as:

$$V(y_s^0, s) = \underset{\{u_t^0\}_{s+1}^T}{Min} \left( ½(y_T^{0'} K_T^0 y_T^0) + \sum_{t=s+1}^{T-1} [½(y_t^{0'} K^0 y_t^0)] \right),$$

(A1)

$$\textit{subject to} \quad y_t^0 = A^0 y_{t-1}^0 + C^0 u_t^0 + G^0 \varepsilon_t^0, \quad t = s+1, \ldots, T, \quad y_s^0 \text{ given},$$

where $y_t^0 = \beta^{(t-1)/2} \begin{pmatrix} \bar{y}_t \\ \bar{u}_t \end{pmatrix}$, $u_t^0 = \beta^{(t-1)/2} \bar{u}_t$, $\varepsilon_t^0 = \beta^{(t-1)/2} \bar{\varepsilon}_t$, and $K^0 = \begin{pmatrix} \bar{K} & J \\ J' & R \end{pmatrix}$,

$K_T^0 = \begin{pmatrix} \bar{K}_T & J_T \\ J_T' & R_T \end{pmatrix}$, $A^0 = \beta^{½} \begin{pmatrix} \bar{A} & 0 \\ 0 & 0 \end{pmatrix}$, $C^0 = \beta^{½} \begin{pmatrix} \bar{C} \\ I \end{pmatrix}$, $G^0 = \beta^{½} \begin{pmatrix} \bar{G} \\ 0 \end{pmatrix}$.

Necessary second-order conditions are that $C^{0'} K^0 C^0$ and $C_T^{0'} K_T^0 C_T^0$ be positive definite.

$K^0$ and $K_T^0$ are positive semi definite and we assume that $K_T^0$ is a scalar multiple of $K^0$.

Dropping the stochastic elements because of the well-known property of certainty equivalence, implies:

$$V(y_s, s) = \underset{\{u_t\}_{s+1}^T}{Min} \left( ½(y_T' K_T y_T) + \sum_{t=s+1}^{T-1} [½(y_t' K y_t)] \right),$$

(A2)

$$\textit{subject to} \quad y_t = A y_{t-1} + C u_t, \quad t = s+1, \ldots, T, \quad y_s \textit{ given},$$

Thus, the LQC problem (A2) determines $u_t$ which is sufficient to obtain the value function. However, to determine the optimal control solution for the original Problem 1, having solved for $u_t = -F_t y_{t-1}$, the relevant optimal control $\bar{u}_t$ can be obtained from:

(A3) $\bar{u}_t = -\beta^{-(t-1)/2} F_t y_{t-1}$,

(A4) $y_t = (A - CF_t) y_{t-1} + \beta^{t/2} \begin{pmatrix} \bar{G} \\ 0 \end{pmatrix} \bar{\varepsilon}_t$.

## Appendix B.1: Proof of Lemma 1

From equations (9) it is straightforward to relate $P_{3t}$ ($= P_{2t}^{/}$) and $P_{4t}$ to $P_{1t}$. The first equation in (9) gives $P_{2t} = -P_{1t} C_1 C_2^{-1}$. Transpose (to produce $P_{3t}$) and substitute into the second equation (noting the symmetry of $P_{1t}$ as follows from the symmetry of $P_t$) which yields $P_{4t} = (C_2^{/})^{-1} C_1^{/} P_{1t} C_1 C_2^{-1}$. Then factor out the $M$ and $M^{/}$ matrices to produce equation (10). To show that $\Phi_t$ is invertible, note from equation (8) that $P_t$ can be written as the product $H_t [I_n - C(C^{/} H_t C)^{-1} C^{/} H_t]$, where the matrix in brackets is idempotent with trace equal to $trace(I_n) - trace(I_k)$ and thus rank $n - k = q$. Hence, since $H_t$ has full rank $n$, $P_t$ has rank $q$ by Sylvester's inequality:

(B1.1) $\text{rank}(X_1) + \text{rank}(X_2) - n \leq \text{rank}(X_1 X_2) \leq \min[\text{rank}(X_1), \text{rank}(X_2)]$,

where $n$ is the number of rows in $X_2$.

Equation (10) then implies that rank ($\Phi_t$) $\geq q$, and since $\Phi_t$ has dimension $q$ it must have full rank. ■

## Appendix B2: Proof of Lemma 2

Post-multiply equation (8) by $H_t^{-1}P_t$ and then use the transpose of equation (9). This yields

$P_t = P_t H_t^{-1} P_t$, so that , interestingly, $H_t^{-1}$ is seen to be a generalized inverse of $P_t$. Next use equation (10) in the right-hand side of this equation and pre-multiply by $\begin{pmatrix} I_q & 0 \end{pmatrix}$ and post-multiply by $\begin{pmatrix} I_q & 0 \end{pmatrix}'$ to pick out the upper left block $P_{1t} \equiv \Phi_t$ of the matrix, yielding:

$$\text{(B2.1)} \quad \Phi_t = \begin{pmatrix} I_q & 0 \end{pmatrix} M \Phi_t M' H_t^{-1} M \Phi_t M' \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \qquad t \le T.$$

Now consider that $\begin{pmatrix} I_q & 0 \end{pmatrix} M = I_q$, and post-multiply equation (13) by $\Phi_t^{-1}(M' H_t^{-1} M)^{-1}$, to obtain equation (12). Positive definiteness follows directly from equation (12) given that $H_t$ is positive definite. ■

## Appendix B3: Proof of Theorem 1

Substitute $P_t = M \Phi_t M'$ from Lemma 1 into equation (7):

$$\text{(B3.1)} \quad H_{t-1} = K + A' M \Phi_t M' A, \qquad t \le T.$$

A standard inversion identity (used later on further occasions) states that given the matrices $X_1$, $X_2$, $X_3$, and $X_4$, with $X_1$ and $X_4$ invertible, we have [Söderström (1994), pp. 156-7]:

$$\text{(B3.2)} \quad (X_1 + X_2 X_4^{-1} X_3)^{-1} = X_1^{-1} - X_1^{-1} X_2 (X_4 + X_3 X_1^{-1} X_2)^{-1} X_3 X_1^{-1}$$

Applying the identity to (B3.1) gives:

$$\text{(B3.3)}\quad H_{t-1}^{-1} = K^{-1} - K^{-1}A'M(\Phi_t^{-1} + M'AK^{-1}A'M)^{-1}M'AK^{-1}.$$

Post-multiplying by $M$ and pre-multiplying by $M'$ yields after applying Lemma 2:

$$\text{(B3.4)}\quad \Phi_{t-1}^{-1} = M'K^{-1}M - M'K^{-1}A'M(\Phi_t^{-1} + M'AK^{-1}A'M)^{-1}M'AK^{-1}M,$$

which is equation (13). $\Phi_T^{-1} = M'K_T^{-1}M$ follows from equation (12) using the fact that $H_T = K_T$ from (4). ■

## Appendix B4: Proof of Theorem 2

From equations (3) and (8) we obtain

$$\text{(B4.1)}\quad CF_t = (I_n - H_t^{-1}P_t)A = (I_n - H_t^{-1}M\,\Phi_t M')A, \qquad t \le T,$$

where the second equality follows from Lemma 1. To obtain the term $H_t^{-1}M$ appearing on the right-hand side of equation (B4.1), we first use equation (B3.3) and the definitions in Theorem 1:

$$\text{(B4.2)}\quad H_{t-1}^{-1}M = K^{-1}M - K^{-1}A'M(\Phi_t^{-1} + B_3)^{-1}B_2, \qquad t \le T.$$

Use the solution of equation (13) for $(\Phi_t^{-1} + B_3)^{-1}$ in equation (B4.2):

$$\text{(B4.3)}\quad H_{t-1}^{-1}M = K^{-1}M - (K^{-1}A'M)\,B_2^{-1'}(B_1 - \Phi_{t-1}^{-1}), \qquad t \le T.$$

Update equation (B4.3) by one period (making it valid for $t \le T - 1$) and substitute into the right side of (B4.1). Pre-multiplying the left and right sides of equation (B4.1) by $(C'C)^{-1}C'$ yields equation (14). ■

**Appendix B5: Proof of Theorem 1***

Consider a generalized inverse of $B_2$ given as: $B_2^I = S\, I_r^q\, Q$. It is easy to check that $B_2^I$ is indeed a generalized inverse of $B_2$ since $B_2\, B_2^I\, B_2 = Q^{-1}\,(I_r^q)^3\, S^{-1} = B_2$ and since $I_r^q$ is an idempotent matrix. Define $\Theta = (I_q - B_2^I B_2)\; S \begin{pmatrix} 0_{r,\,q-r} \\ I_{q-r} \end{pmatrix}$. Then by design $B_2\,\Theta = 0_{q,\,q-r}$ and straightforward multiplication shows that:

$$\text{(B5.1)} \quad \Theta = \left(I_q - S\,(I_r^q)^2\, S^{-1}\right) S \begin{pmatrix} 0_{r,\,q-r} \\ I_{q-r} \end{pmatrix} = S\left(I_q - I_r^q\right)\begin{pmatrix} 0_{r,\,q-r} \\ I_{q-r} \end{pmatrix} = \begin{pmatrix} S_2 \\ S_4 \end{pmatrix} .$$

Since $B_2\,\Theta = 0$ it follows from equation (14) updated one period that:

$$\text{(B5.2)} \quad \left(\Phi_t^{-1} - B_1\right)\Theta = \left(\Phi_t^{-1} - B_1\right)\begin{pmatrix} S_2 \\ S_4 \end{pmatrix} = 0_{q,\,q-r}, \qquad t \le T-1 .$$

Partition $\Phi_t^{-1}$ and $B_1$ according to $S$ and extract the $r$ x $r$ upper left block of $\Phi_t^{-1}$ as $Z'\,\Phi_t^{-1}\,Z$.

Take $S_4$ as invertible[6] and solve equations (B5.2) similarly to equation (9). This yields:

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[6] If $S_4$ is not invertible we can always rearrange the rows of $S$ by pre-multiplying by some permutation matrix $J$ such that $\hat{S} \equiv J\,S$ with $\hat{S}_4$ invertible. In this case transform all $B_i$ to obtain $J'\,B_i\,J$. In order for equation (14) to continue to hold we must also transform $\Phi_t^{-1}$ to $J'\,\Phi_t^{-1}\,J$ and in order for the procedure to work we must also set $\hat{Q} \equiv Q\,J'$. Then we can proceed as above. We then obtain $J'\,(\Phi_t^{-1} - B_1)\,J$ in equation (B5.3) but can recover $\Phi_t^{-1}$ by inverting the known $J$ matrix.

(B5.3) $\quad \Phi_t^{-1} - B_1 = M^* (\Phi_t^{*-1} - B_{11}) M^{*\prime}, \quad M^* = \begin{pmatrix} I_r \\ -S_4^{-1\prime} S_2^{\prime} \end{pmatrix}, \qquad t \leq T - 1,$

where $\Phi_t^{*-1} \equiv Z'\Phi_t^{-1} Z$ represents the upper left block of $\Phi_t^{-1}$ and $B_{11} \equiv Z' B_1 Z$ represents the upper left block of $B_1$. As $\Phi_t^{-1}$ and $B_1$ are positive definite, so are $\Phi_t^{*-1}$ and $B_{11}$.

To obtain the dynamics of $\Phi_t^{*-1} \equiv Z' \Phi_t^{-1} Z$, post-multiply equation (14) by $Z$ and pre-multiply by $Z'$, and use equation (B5.3) on the right side to produce:

(B5.4) $\quad \Phi_{t-1}^{*-1} - B_{11} = -Z' B_2' [M^* (\Phi_t^{*-1} - B_{11}) M^{*\prime} + (B_1 + B_3)]^{-1} B_2 Z, \qquad t \leq T - 1.$

To manipulate equation (B5.4), consider that the term in brackets is positive definite (as it equals the sum of a positive definite matrix $\Phi_t^{-1}$ and a positive semi-definite matrix $B_3$). Further, $B_2 Z$ has full column rank $r$: $\quad B_2 Z = Q^{-1} I_r^q S^{-1} Z, \quad S^{-1} \equiv \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix}$. Thus $I_r^q S^{-1} Z = \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix}$,

where $\operatorname{rank} \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix} = \operatorname{rank}(\Sigma_1)$, so $\operatorname{rank}(B_2 Z) = \operatorname{rank}(\Sigma_1)$. From footnote 4 we can assume without loss of generality that $S_4$ has full rank $q - r$. We know that $\Sigma_1 = (S_1 - S_2 S_4^{-1} S_3)^{-1}$ [Anderson and Moore (1990), p. 349] because $S_4^{-1}$ exists and because $|S_4| \cdot |S_1 - S_2 S_4^{-1} S_3| = |S| \neq 0$ so $|S_1 - S_2 S_4^{-1} S_3| \neq 0$ [Söderström (1994), p. 162]. Thus $\Sigma_1$ has full rank $r$, so that $B_2 Z$ has full column rank. This fact, together with the positive definiteness of the term in brackets, establishes that the left-hand side of equation (B5.4) is negative definite and thus $\Phi_t^{*\,-1} - B_{11}$ is invertible. We can now use the inversion identity (B3.2) to rewrite the term in brackets, since the relevant inverses exist:

$$\text{(B5.5)} \quad \Phi_{t-1}^{*\,-1} = Z'[B_1 - B_2'(B_1 + B_3)^{-1}B_2]Z + Z'B_2'(B_1 + B_3)^{-1}M^* \cdot$$

$$[M^{*\prime}(B_1 + B_3)^{-1}M^* + (\Phi_t^{*\,-1} - B_{11})^{-1}]^{-1} M^{*\prime}(B_1 + B_3)^{-1} B_2 Z\,, \qquad t \le T - 1.$$

Again use the inversion identity to reformulate the second expression in brackets:

$$\text{(B5.6)} \quad \Phi_{t-1}^{*\,-1} = Z'[B_1 - B_2'(B_1 + B_3)^{-1}B_2]Z + Z'B_2'(B_1 + B_3)^{-1}M^* \Big\{ [M^{*\prime}(B_1 + B_3)^{-1}M^*]^{-1}$$

$$- [M^{*\prime}(B_1 + B_3)^{-1}M^*]^{-1} \{ \Phi_t^{*\,-1} - Z'B_1Z + [M^{*\prime}(B_1 + B_3)^{-1}M^*]^{-1} \}^{-1} \cdot$$

$$[M^{*\prime}(B_1 + B_3)^{-1}M^*]^{-1} \Big\} M^{*\prime}(B_1 + B_3)^{-1} B_2 Z\,, \qquad t \le T - 1.$$

Note that the inverse of the term in small braces exists by the invertibility of the second term in brackets in equation (B5.5), since, when $[X + Y^{-1}]^{-1}$ exists, so does $[Y + X^{-1}]^{-1}$ with $Y \equiv \Phi_t^{*\,-1} - B_{11}$ and $X \equiv M^{*\prime}(B_1 + B_3)^{-1}M^*$ both invertible.

Equation (B5.6) directly yields equation (15) in Theorem 1*. The initial condition $\Phi_{T-1}^{*\,-1}$ is obtained from using equation (14) in $\Phi_{T-1}^{*\,-1} \equiv Z'\Phi_{T-1}^{-1}Z$. ■

An example in which $B_2^*$ is singular (after a single application of the reduction in Theorem 1*) is as follows:

$$K = I_4\,, \quad C = \begin{pmatrix} 0_{2,2} \\ I_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 \\ -0.5 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $B_2 = A_1$ and so $rank\,(B_2) = 1$ and hence $B_2^*$ is a scalar. Then

$$Q = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -0.5 \\ 0 & -1 \end{pmatrix}, \text{ so } M^* = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}. \text{ Therefore } M^{*\prime}(B_1 + B_3)^{-1} B_2 Z = 0 \text{ so}$$

that $B_2^* = 0$; hence $rank\ B_2^* = 0 < 1$.

## Appendix C: MATLAB Program and Examples

The programs and examples are available from http://www.be.wvu.edu/divecon/econ/balvers/riccatimatlab.htm .

A MATLAB program that follows exactly the approach in this paper is RedFunction.m. A more practical program RedMainFin.m includes also the transformations to convert to minimal order and allows for a singular state cost matrix. This program calls a simple Riccati iteration procedure to generate the feedback matrix and Riccati kernel for each point in finite time. Each of the transformations can also be called separately as functions: RedCon.m, RedObs.m, RedK.m, RedA.m. Finally, the program RedMainInf.m uses our reduction but then solves for the algebraic Riccati and feedback matrices (using either the OLRP.m program of Ljungqvist and Sargent or the DARE.m program from MATLAB's control toolbox).

The programs are designed to compare the results from alternative approaches, but can easily be amended. Various examples are available (three examples developed for this paper, two examples from Amman and Neudecker, 1997, and two examples from Ljungqvist and Sargent, 2000).

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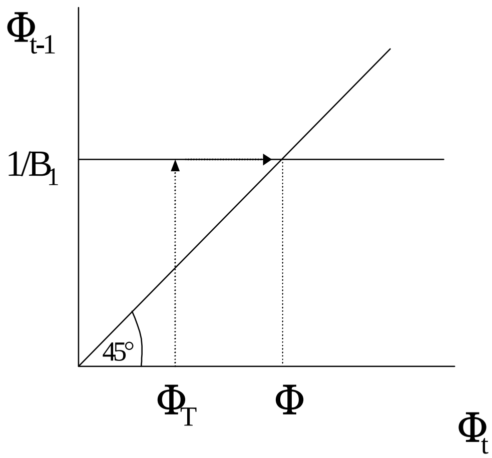


**Fig. 1(a):** $B_2 = 0$

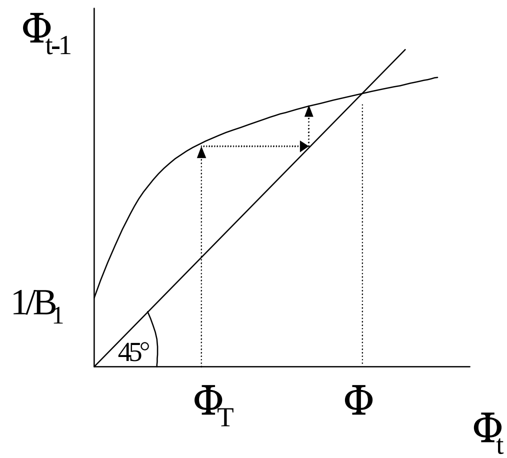


**Fig. 1(b):** $B_2 \neq 0, \;\; B_1 B_3 > B_2^2$

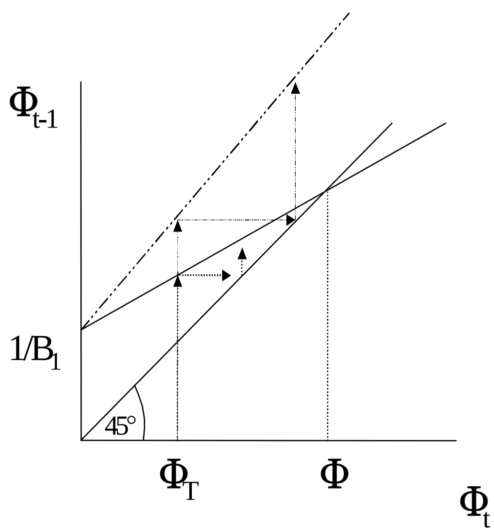


**Fig. 1(c):** $B_2 \neq 0, \;\; B_1 B_3 = B_2^2$

## Reply to Associate Editor

1. We have focused the exposition primarily on finite horizon LQC problems, although in our framework infinite horizon problems follow naturally from these. We illustrate our solution algorithm using an interesting rational expectations model, which is most naturally specified with an infinite horizon.

   MATLAB code for the transformation of any problem with constant coefficients to our canonical form, and for the Riccati reduction of our paper, is available from our website as cited in the paper.

2. We do not report results from numerical experiments on the speed of the algorithm, and we have eliminated our previous misleading references to algorithmic speed. The point about speed of implementation is simply that any numerical solution technique that could be applied to the original Riccati equation can be implemented faster on our Riccati kernel of reduced size.

3. Our MATLAB program (REDMAINFIN.M) allows one to start with any positive semi-definite $K$ matrix. However, in the text, to avoid substantial further complications as a result of working with generalized inverses, we maintain the assumption of an invertible state cost matrix $K$. Footnote 4 on p. 6 discussed this issue. This footnote also clarifies the assumptions on the $C$ matrix.

4. Beyond minimality, which we discuss more carefully in the present version, the literature contains no work on Riccati matrix reduction beyond Mitchell (JEDC, 2000), which was already cited in the previous version.

5. As you requested we start with a formulation of the problem that incorporates control costs explicitly in the objective.

6. We have tried to write the current version as tightly as possible. In particular, we have removed two corollaries, two theorems, and the discussion of Kalman filtering.

7. The current version keeps the use of control theory jargon to a minimum.

## Reply to Referees

We thank all three referees for their thoughtful and detailed comments. Based on these comments we have made the following revisions.

### Referee I

1. We have clarified in section 2.3 that our procedure generally gives a net reduction in the size of the Riccati matrix.

2/3. We do not report results from numerical experiments on the speed of the algorithm, and we have eliminated our previous misleading references to algorithmic speed. The point about speed of implementation is simply that any numerical solution technique that could be applied to the original Riccati equation can be implemented faster on our Riccati kernel of reduced size. We acknowledge the issue of sparsity on p.23.

4. We have replaced our unnecessarily restrictive assumptions on the matrices with the least restrictive assumptions possible. Our MATLAB program (REDMAINFIN.M) allows one to start with any positive semi-definite *K* matrix. However, in the text, to avoid substantial complications to an already complex derivation as a result of working with generalized inverses, we maintain the assumption of an invertible state cost matrix *K*. Footnote 4 on p. 6 discussed this issue. This footnote also clarifies the assumptions on the *C* matrix.

5. Beyond minimality, which we discuss more carefully in the present version, the literature contains no work on Riccati matrix reduction beyond Mitchell (JEDC, 2000), which was already cited in the previous version. Since the focus of the paper is on identifying the kernel of the dynamic structure of Riccati equations, and not on numerical analysis, we have not extensively cited the literature on the latter although we refer to Patel, Verhagen, and van Dooren (1994) on p. 23.

6. For finite horizon problems which must be solved iteratively our technique reduces the dimension of the recursions. For infinite horizon problems our technique reduces the size of the algebraic Riccati equation, whatever solution technique may be chosen.

*More Detailed Comments*

1. While we appreciate the desire for brevity in the choice of references, we have retained the indicated references because we feel that these references motivate the discussion of LQC problems from an economics perspective.

2. We have deleted the indicated sentence.

3. In p.1 paragraph 2 we have clarified the benefits of the paper.

4. We have clarified the indicated passage.

5. We have dropped the adjective "simple."

6. We have eliminated the unclear run-on sentence.

7. We have deleted the mention of alternative terminology in the Kalman filtering context, because we have almost completely eliminated mention of this context from the paper.

8. We now define all notation before stating its properties.

9/11. Our paper replaces the old equation (3) with a reduced version which in the infinite horizon case can be solved with any numerically stable approach.

10. We now show in Appendix A how to convert a general problem formulation with invertible $K$ matrix into our canonical form. Our MATLAB program does so for singular $K$ matrix as well.

12. Both the inversion identity and Sylvester's inequality are no longer in footnotes and are stated explicitly in the Appendix.

13. Our procedure certainly does not assume that $A$ is nonsingular, and indeed the extent of the reduction is generally greater when $A$ is singular.

14/15/16. Corollaries 1.1 and 1.2 have been removed from the paper, but we have chosen to retain Theorem 2 because it helps to illuminate the structure of the problem.

17. We have clarified in the introduction what the purpose of the paper is.

18. We have deleted Theorems 3 and 3*.

19. We have substantially shortened section 3.

20. We have removed the mention of eigenvalue computations or Jordan or Schur decompositions.

21. We have removed this paragraph.

**Referee II**

*Overall assessment*

We point out that the main contribution of the paper does not lie in the speed of iteration in the reduced Riccati equation. See point 2/3 in the reply to Referee 1. We discuss computational and non-computational advantages on pp. 22-24, where we also compare our approach to that of Binder and Pesaran (2000).

*Detailed comments*

1. Our formulation with $K$ constant prior to time $T$, but possibly differing at $T$, is a very widely used formulation. In addition, while our approach does not preclude putting a high cost on deviations of the final state vector, our approach with no end point restriction is very widely used.

2. Footnote 4 on p.6 points out that the case in which $C$ has less than full column rank is uninteresting because there are superfluous controls which can be eliminated without affecting the optimal value of the loss function. This is further necessary for second-order conditions to hold.

3. We have indicated that the main point of the paper is to explore the nature of the kernel of the Riccati dynamics and not to explore the numerical calculation expense. (See pp. 22-24 in particular).

4. We have removed the section on Kalman filtering.

5. We no longer refer to numerical instabilities, which are not the point of the paper.

**Referee III**

1. We have emphasized that our reduction applies when the system is minimal. See in particular pp.4 and 7, and the example. In addition, our MATLAB program allows the problem to be converted to minimal form.

2. While we have retained our previous notation, we have adapted the formulation with explicit control costs.

3. We remove the inadvertent implication that the standard solution originated with Chow.

4. We discuss that our approach can reduce even a minimal problem and provide an explicit example in the text and several other examples with our MATLAB program. However, we do not explicitly impose those restrictions in the text as these restrictions would not affect the theoretical analysis and can easily be assumed by the reader without affecting the basic results. See p.7.

5. We show in Appendix A how to convert a general problem formulation with invertible $K$ matrix into our canonical form. Our MATLAB program does so for singular $K$ matrix as well. We discuss in footnote 4 the cost of assuming invertible $K$ in the text in terms of complicating an already difficult paper by working with generalized inverses.

6. We have removed the section on Kalman filtering as you suggest.

7. In our conclusion we raise but do not resolve the issue of whether the matrices to be inverted are well conditioned.