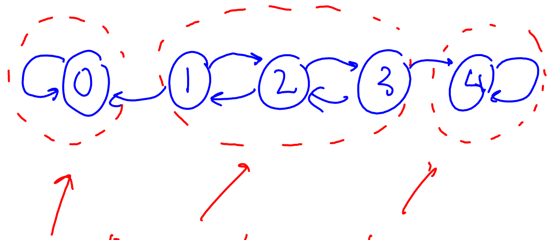
2013-11-18

Monday, November 18, 2013
12:27 PM

Make-up class on Dec. 9? ($12^{30}$-$3^{30}$)

Ex.



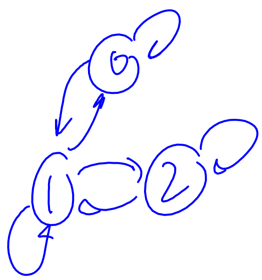
Equivalence classes

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & & & & \\ 1 & \times & & \times & & \\ 2 & & \times & & \times & \\ 3 & & & \times & & \times \\ 4 & & & & & 1 \end{array}$$

Def If a MC has all its states belonging to one equivalence class it is said to be irreducible; i.e., if all states communicate

(GOOD Property)

Ex



Irreducible

Ex. (s, S) Inv.
1, 3

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \times & \times & \times & \times \\ 1 & \times & \times & 0 & 0 \\ 2 & \times & \times & \times & 0 \\ 3 & \times & \times & \times & \times \end{array}$$

Maple worksheet

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/documents/Irreducibility-Algorithm.pdf

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Reducible-MC-Simpler-N10.mw

(i) Periodicity

Def. The period of a state $i$, $d(i)$, is g.c.d.

Def. The period of a state $i$, $d(i)$, is g.c.d. of all integers $n \geq 1$ for which $P_{ii}^{(n)} > 0$.

When period is 1, the state is aperiodic ("GOOD")

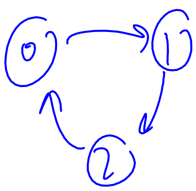
Ex. $2, 4, 6, 8, \ldots$ : gcd $= 2$

Ex. $56, 12$. gcd $= 4$ igcd(56,12) = 4

Ex. $P_{ii}^{(1)} > 0$, $P_{ii}^{(2)} > 0$, $P_{ii}^{(3)} = 0$, $P_{ii}^{(4)} > 0$, $\ldots$

1 2 4 : gcd = 1

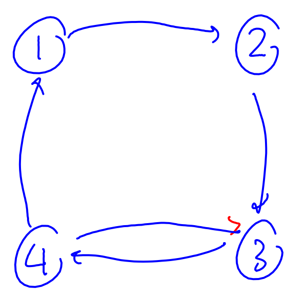
Ex.



$d(i) = 3$, $i = 0, 1, 2$

Ex. Maple method (Irreducible)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array} \begin{array}{c} ? \\ .33 \\ .16 \\ .16 \\ .33 \end{array}$$



$\lim_{n \to \infty} P^{(n)}$ doesn't exist. won't converge

$P_{33}^{(1)} = 0$ $P_{33}^{(2)} > 0$ $P_{33}^{(3)} = 0$, $P_{33}^{(4)} > 0$, -

2 4

$d(3)=2$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Period.mw

Theorem Periodicity is a class property, i.e., if state $i$ has period $d(i)$ and $i \leftrightarrow j$, then $d(i)=d(j)$.

## (ii) Recurrence

Let $T_{ij} = \min\{n : X_n = j \mid X_0 = i\}$ FPT

$f_{ii}^{(n)} = \Pr\{\text{first transition into } i \text{ at } n \mid \text{start in } i\}$

$f_{ii}^{(0)} = 0$

$f_{ii}^{(1)} = p_{ii}$

$f_{ii}^{(n)} = \Pr\{X_n = i,\ X_k \neq i,\ k=1,2,\ldots,n-1 \mid X_0 = i\},\ n \geq 2$

[Kao, p.169. $f_{ij}^{(n)} = \sum_{k \neq j} p_{ik} f_{kj}^{(n-1)},\ n \geq 2$]

Define $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$ : $\Pr\{\text{ever going to } i \mid \text{start in } i\}$

$= f_{ii}^{(1)} + f_{ii}^{(2)} + f_{ii}^{(3)} + \cdots$

**Def** State $i$ is recurrent (persistent) if $f_{ii} = 1$; i.e., ultimate return to $i$ is certain; transient if $f_{ii} < 1$

**Note:** Usually difficult to test $f_{ii}$. More useful result ↓

**Theorem** State $i$ is recurrent $\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$

i.e.,

$f_{ii} = 1 \iff \sum P_{ii}^{(n)} = \infty$ recurrent

$f_{ii} < 1 \iff \sum P_{ii}^{(n)} < \infty$ transient

**Note.** For a recurrent state $i$

$f_{ii}^{(1)}, f_{ii}^{(2)}, f_{ii}^{(3)}, \ldots$

$f_{ii} = \sum f_{ii}^{(n)} = 1$

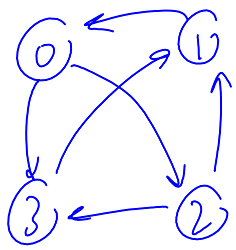

**Corollary** If $i$ is transient, it will be visited a finite # times

**Corollary** In a finite MC, not all states can be transient ($\Pi e \overset{?}{=} 0$)

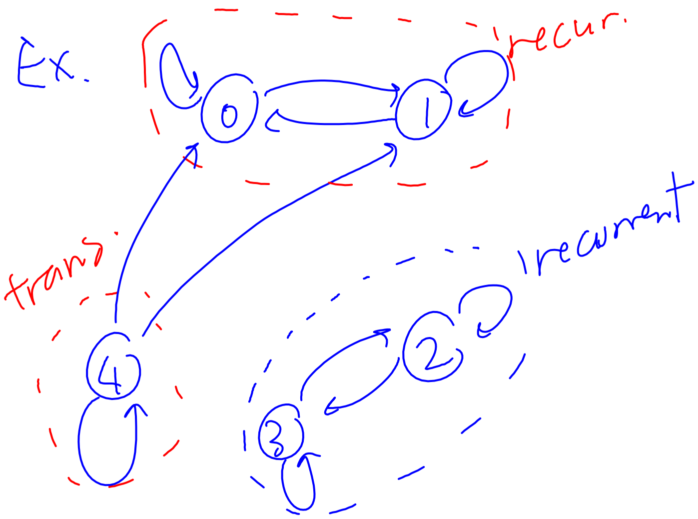
**Theorem** Recurrence is a class property, i.e., if

(GOOD prop·ty) $i$ is recurrent and $i \leftrightarrow j$, then $j$ is also recurrent

Ex.

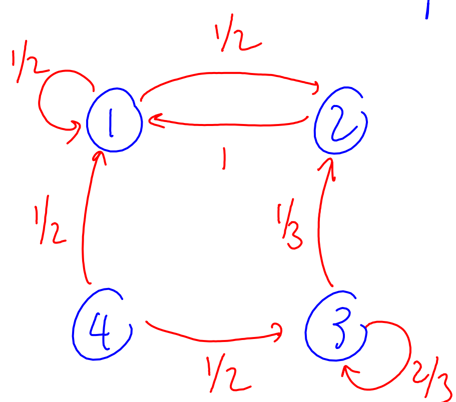



Ex.




Ex. Transient & recurrent states (by direct computation)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$




$$\left.\begin{array}{ll} ④ & f_{44}^{(n)} = 0 < 1 \quad \forall n \\ ③ & f_{33}^{(1)} = \frac{2}{3}, \quad f_{33}^{(n)} = 0 \end{array}\right\} \quad \left.\begin{array}{l} f_{44} < 1 \\ f_{33} < 1 \end{array}\right\} \text{transient}$$

② $f_{22}^{(1)}=0,\quad f_{22}^{(2)}=1\cdot\frac{1}{2}=\frac{1}{2},\quad f_{22}^{(3)}=1\cdot\frac{1}{2}\cdot\frac{1}{2},\ \cdots\ f_{22}^{(n)}=\frac{1}{2^{n-1}},\ n\geq 2$

$$f_{22}=\sum_{1}^{\infty} f_{22}^{(n)}=1$$

① + ② recurrent

① $f_{11}^{(1)}=\frac{1}{2},\quad f_{11}^{(2)}=\frac{1}{2}\cdot 1=\frac{1}{2},\quad f_{11}=\sum f_{11}^{(n)}=1$

## d) Limit Theorems

Def Mean recurrence time (FPT) $\mu_{jj}$ is the E(# transitions needed to return to j) for a recurrent state j, where

$$(f_{jj}=1\Rightarrow)\quad \mu_{jj}=\sum_{n\geq 1}^{\infty} n f_{jj}^{(n)} \begin{cases} <\infty & : \text{positive-recurrent } j \ \text{(GOOD)} \\ =\infty & : \text{null-recurrent } j \ \text{(not good)} \end{cases}$$

Theorem Positive (or null) recurrence is class property.

Ex. Above problem (States 1 & 2)

$$\mu_{11}=\sum_{n=1}^{\infty} n f_{11}^{(n)}=1\cdot\frac{1}{2}+2\cdot\frac{1}{2}=\frac{3}{2}<\infty$$

$$\mu_{22}=\sum_{n=1}^{\infty} n f_{22}^{(n)}=\sum_{n=2}^{\infty} n\,\frac{1}{2^{n-1}}=3<\infty$$

Calculus (or Maple)

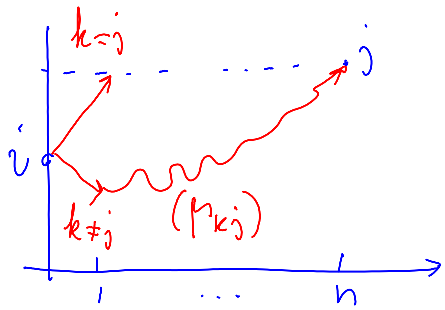
#3

$$\mu_{22} = 3 \ , \quad \pi_2 = \frac{1}{\mu_{22}} = \frac{1}{3}$$

Simpler formula for $\mu_{ij}$?

Recall: $T_{ij} = \min\{n : X_n = j \mid X_0 = i\}$ : FPT $i \to j$

$$\mu_{ij} = E(T_{ij}) = \sum \underbrace{E(T_{ij} \mid X_1 = k, X_0 = i)} \cdot \Pr\{X_1 = k \mid X_0 = i\}$$




$$E(T_{ij} \mid X_1 = k, X_0 = i) = \begin{cases} 1 \ , & k = j \\ 1 + \mu_{kj} \ , & k \neq j \end{cases}$$

$$\therefore \quad \mu_{ij} = 1 \cdot p_{ij} + \sum_{k \neq j} p_{ik}(1 + \mu_{kj})$$

$$\Rightarrow \quad \mu_{ij} = 1 + \sum_{k \neq j} p_{ik}\mu_{kj} \ , \quad \forall i, j \in I$$

Ex. Rainfall

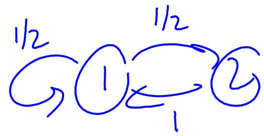
$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{array} \qquad \begin{array}{l} \pi_0 = 4/7 \\ \pi_1 = 3/7 \end{array}$$

$$\left.\begin{aligned} \mu_{00} &= 1 + p_{01}\mu_{10} \\ \mu_{01} &= 1 + p_{00}\mu_{01} \\ \mu_{10} &= 1 + p_{11}\mu_{10} \\ \mu_{11} &= 1 + p_{10}\mu_{01} \end{aligned}\right\} \Rightarrow \begin{aligned} \mu_{00} &= 7/4 = 1/\pi_0 \\ \mu_{01} &= 10/3 \\ \mu_{10} &= 5/2 \\ \mu_{11} &= 7/3 = 1/\pi_1 \end{aligned}$$

Ex. Our problem

$$P = \begin{array}{c c} 1 & \\ 2 & \end{array}\!\!\begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{recurrent } 1, 2$$

$\mu_{11} = 3/2$ $\quad\mu_{12} = 2$




$\mu_{21} = 1$ $\quad\mu_{22} = 3$

Ex. $(s, S)$ w/ Maple

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/muij.mw

{mu[1, 1] = 3.499253474, mu[1, 2] = 3.973665565, mu[1, 3] = 3.509359015, mu[1, 4] = 6.012172114,
mu[2, 1] = 1.582028160, mu[2, 2] = 3.511754003, mu[2, 3] = 5.091387175, mu[2, 4] = 7.594200274,
mu[3, 1] = 2.502813099, mu[3, 2] = 3.242908467, mu[3, 3] = 3.800293993, mu[3, 4] = 8.514985213,
mu[4, 1] = 3.499253474, mu[4, 2] = 3.973665565, mu[4, 3] = 3.509359015, mu[4, 4] = 6.012172114}

<u>Lemma</u> (Discrete version of KRT)

Given $\{f_n\}$ such that $f_0 = 0$, $f_n \geq 0$, $\sum f_n = 1$, and the gcd of those $n$ for which $f_n > 0$ is $d$ ($\geq 1$), a second sequence $\{u_n\}$ is defined as

$u_0 = 1$, $f_0 = 0$,

$f_n = Pr(\text{lifetime} = n)$

$u_0 = 1,\ f_0 = 0,$

$$u_n = f_n + \sum_{k=0}^{n-1} f_k u_{n-k} \quad (n \geq 1)$$

$f_n = \Pr(\text{lifetime} = n)$

$u_n = \Pr(\text{a renewal in } n)$

Then
$$\lim_{n\to\infty} u_{nd} = \begin{cases} d/\mu, & \text{if } \mu = \sum_{n=1}^{\infty} n f_n < \infty \\ 0 & \text{if } \mu = \infty \end{cases}$$

$m(t) = f(t) + \int_0^t f(x) m(t-x)\,dx$

$\to \frac{1}{\mu}$ (KRT)

$u_3 = f_3 + f_2 u_1 + f_1 u_2$


Theorem. Consider;

(Prabhu, p.47)

| | | Periodicity | |
|---|---|---|---|
| | | Aperiodic | Periodic: $d(j) = d$ |
| Transient | | $p_{jj}^{(n)} \to 0$ | |
| Recurrent | null-rec | $p_{jj}^{(n)} \to 0$ | |
| | Pos.-rec | $p_{jj}^{(n)} \to \frac{1}{\mu_{jj}}$ | $p_{jj}^{(nd)} \to \frac{d}{\mu_{jj}}$ |

Ex. Above problem

① & ② aperiodic

$$\lim_{n\to\infty} P_{11}^{(n)} = \frac{1}{\mu_{11}} = \frac{1}{3/2} = \frac{2}{3} = \pi_1$$

$$\lim_{n\to\infty} P_{22}^{(n)} = \frac{1}{\mu_{22}} = \frac{1}{3} = \pi_2$$

Def An ① aperiodic, ② pos. recurrent state is called $\underbrace{\text{ergodic}}$

ergy (energy)   hodos: path

Theorem. Consider an irreducible MC. This MC belongs to one of three classes below

| | Periodicity | |
|---|---|---|
| | Aperiodic | Periodic |
| Transient<br>Recurrent-null | $p_{ij}^{(n)} \to 0$ & $\pi_j = 0$ | |
| 〃 -Pos. | $p_{ij}^{(n)} \to \pi_j > 0$ unique from $\pi = \pi P$, $\pi e = 1$ | $p_{ij}^{(n)} \to ?$ $\{\pi_j\}$: long-run fraction |