2013-10-21

Monday, October 21, 2013

11:17 AM

General results

Q: what happens to $g(t)$ as $t \to \infty$

A: Key Renewal Theorem

- $h(t)$ directly Riemann integrable (DRI)
  - $h(t) \geq 0$
  - $h(t)$ non-increasing
  - $\int_0^\infty h(t)\,dt < \infty$

Under these conditions, for

$$g(t) = h(t) + \int_0^t h(t-x)m(x)dx$$

$$\Rightarrow \quad \lim_{t \to \infty} g(t) = \frac{1}{\mu} \int_0^\infty h(t)\,dt$$

Ex. Equipment

$$g(t) = \underset{\uparrow \text{DRI}}{f(t)} + \int_0^t f(t-x)m(x)dx$$

So, $\displaystyle \lim_{t \to \infty} g(t) = \frac{1}{\mu} \int_0^\infty f(t)\,dt = \frac{1}{\mu}$

In our case, $\mu = \frac{2}{3}$, $\displaystyle \lim_{t \to \infty} g(t) = \frac{3}{2}$

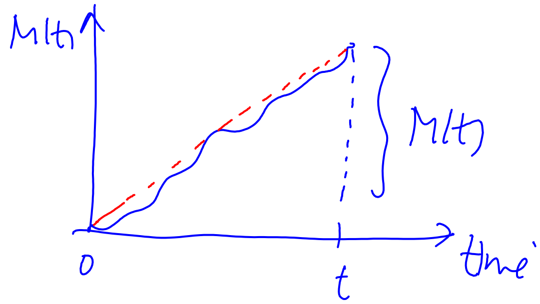
Ex. Renewal eq'n

$$M(t) = F(t) + \int_0^t F(t-x)\, m(x)\, dx$$

$\lim_{t\to\infty} M(t) = ?$ KRT not applicable since $F(t)$ is not DRI

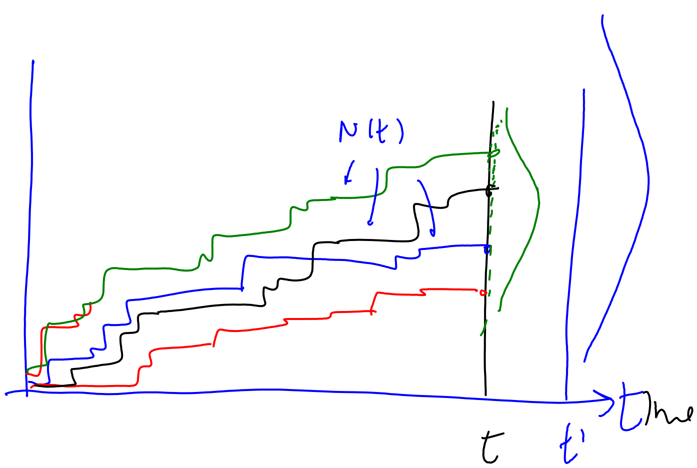
Theorem Elementary renewal theorem

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$$



Theorem. Law of large numbers for renewal process

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{w.p. } 1$$



Theorem Asymptotic distribution of $N(t)$

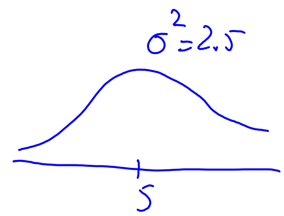
$$N(t) \sim \text{Normal}\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right) \qquad \sigma^2 = Var(X), \quad \mu = E(X)$$

/var

as $t \to \infty$

Ex. $Erl(2, 1)$ (n, λ), $\mu = 2$, $\sigma^2 = 2$

$t = 10$: $N(10) \sim \text{Normal}(5, 2.5)$



$t = 20$: $N(20) \sim \text{Normal}(10, 5)$

Proof (Sketch)

$$M(t) \simeq E(N(t)) = \frac{t}{\mu}$$

$$Var(N(t)) \simeq \frac{\sigma^2}{\mu^3} t$$

Show $$\Pr\left\{\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} \le y\right\} = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

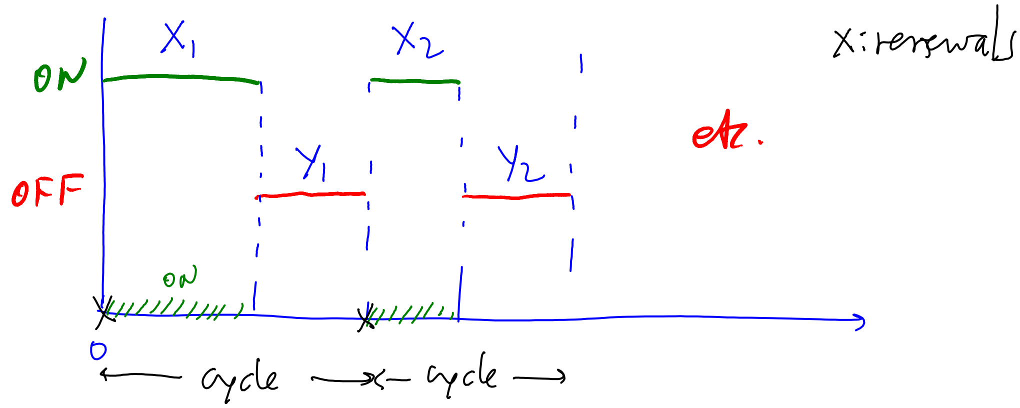
Use $\{N(t) \le n\} \Leftrightarrow \{S_n \ge t\}$

" CLT

$$\Rightarrow \Pr\left\{Z \ge -\frac{y}{\sqrt{1 + \frac{y\sigma}{\cdots}}}\right\} = \Pr\{Z \ge -y\} = \Pr\{Z \le y\}$$

$\left(\sqrt{1+\frac{y_0}{\sqrt{t+s}}}\right) \to 1$ as $t \to \infty$

c) Alternating renewal process



(X_n, Y_n), $n \geq 1$ are iid, but X_n & Y_n may be dependent

X_n has F : ON

Y_n " G : OFF

Cycle $Z_n = X_n + Y_n$ " H : (Initially ON)

Define $P(t) = Pr\{\text{system is ON at } t\}$

<u>Theorem</u> If $E(X_n+Y_n) < \infty$, then

$$\lim_{t\to\infty} P(t) = \frac{E(X_n)}{E(X_n) + E(Y_n)}$$

ON ⇒ 3

OFF ⇒ 1

$\frac{3}{4}$

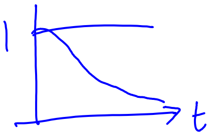
Proof (Sketch): Condition on time of <u>last</u> renewal before t

$$\Rightarrow \quad P(t) = \bar{F}(t) + \int^t \bar{F}(t-r)\, \overbrace{dM(r)}^{m(r)dr}$$

$$\Rightarrow \quad P(t) = \bar{F}(t) + \int_0^t \bar{F}(t-r)\, \underbrace{\widetilde{dM(r)}}_{}$$

for cycle $(X+Y)$

$$\int_0^\infty (1-F(t))\,dt = E(X)$$



$$\lim_{t\to\infty} P(t) = \frac{1}{\mu_H} \underbrace{\int_0^\infty \bar{F}(t)\,dt}_{E(X)} = \frac{E(X)}{E(X)+E(Y)} = Pr(ON)$$

$N \quad n$

$N \leq 20n$

Similarly, $Pr(OFF) = \dfrac{E(Y)}{E(X)+E(Y)}$

Ex. (s,S) Inventory Policy (Continuous review)

Customer arrivals renewal process with F
(X_i: interarrival time)

" demands $Y_1, Y_2, \ldots$ are iid with G

Policy used

$X(t)$: inventory level at t $\qquad X(t) = x$

$S = 10$

$s = 4$

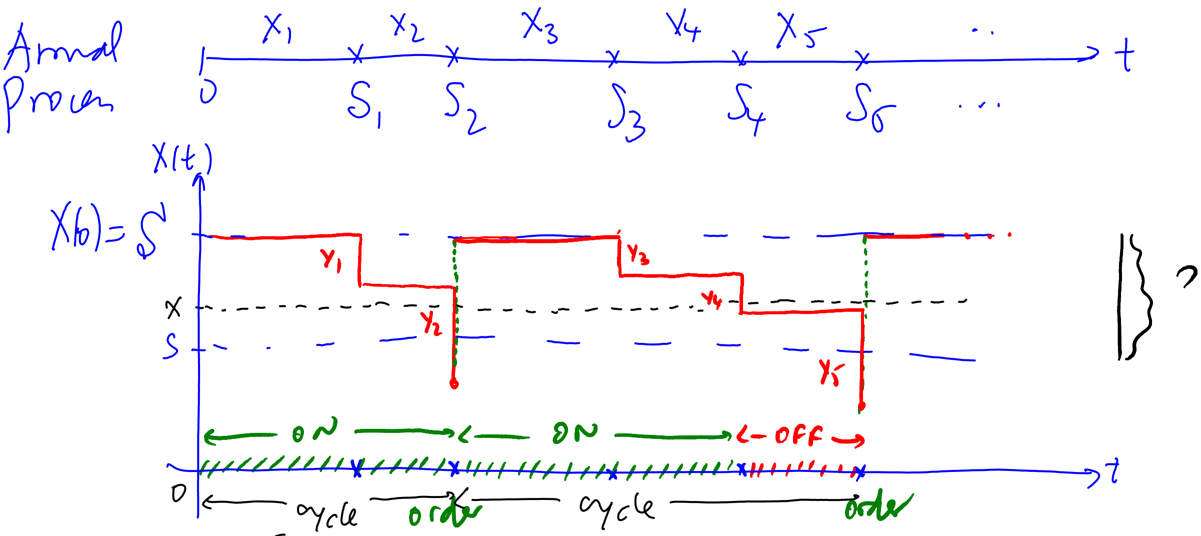
$x = 3$

$\left.\begin{matrix} s \\ S \end{matrix}\right\}$ parameters

$$\text{Order quant} = \begin{cases} S - x & \text{if } x < s \\ 0 & x \geq s \end{cases}$$

Delivery instantaneous

Delivery instantaneous



Find $\Pr\{X(t) \geq x\} = ?$ $\quad s \leq x \leq S$

as $\lim_{t\to\infty}$

$X(t) \geq x$: ON

$$\lim_{t\to\infty} \Pr\{\text{ON at } t\} = \lim_{t\to\infty} \Pr\{X(t) \geq x\}$$

$$= \frac{E(\text{ON})}{E(\text{cycle})} = \frac{E(\text{time inv} \geq x \text{ in a cycle})}{E(\text{time of cycle})}$$

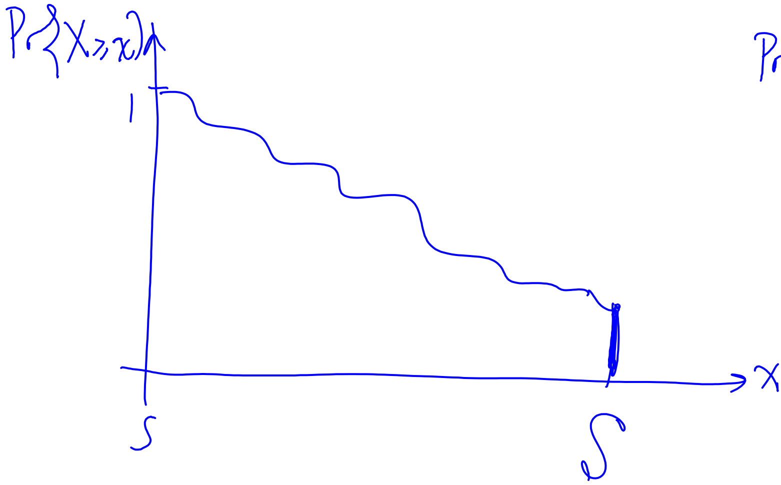
Skipping some developments (Lee, Ross)

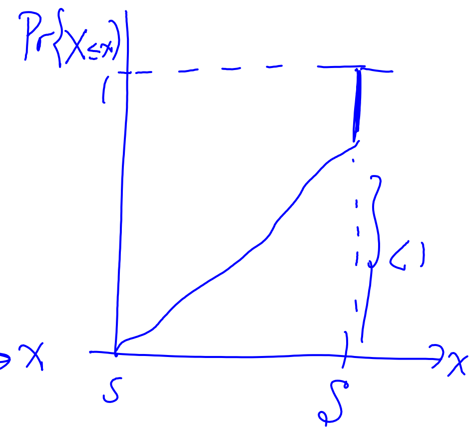
$$\underbrace{\Pr\{X \geq x\} = \lim_{t\to\infty} \Pr\{X(t) \geq x\}} = \frac{1+M_G(S-x)}{1+M_G(S-s)}, \quad s \leq x \leq S$$

DSV ? $\quad 1+M_G(S-s)$

$$\Pr\{X \geq s\} = \frac{\quad}{1+M_G(S-s)} \leq 1$$

$$\Pr\{X \geq S\} = \Pr\{X = S\} = \frac{1}{1+M_G(S-s)} > 0$$



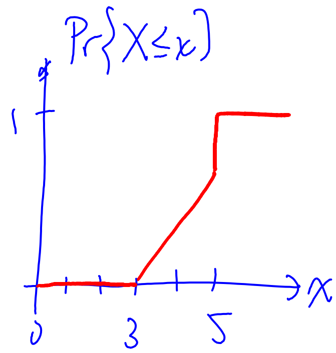


Ex. $Y \sim \exp(1)$, $\lambda = 1$, $M_G(x) = x$

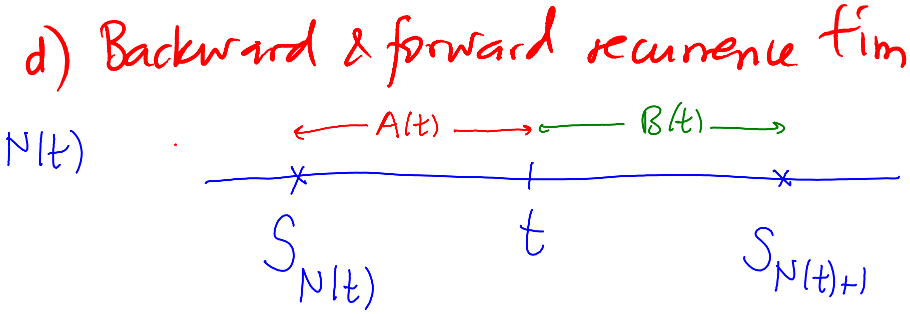
$S = 5$, $s = 3$, $S - s = 2$

$\therefore$ $\Pr\{X = 5\} = \frac{1}{1+2} = \frac{1}{3}$

$\Pr\{X \leq x\} = \frac{1}{3}(x-3)$, $3 \leq x < 5$



d) Backward & forward recurrence times



- Backward recurrence
- Forward recurrence

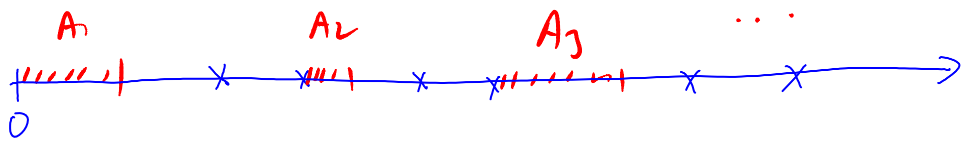
- Current age
- Deficit

- Remaining life
- Excess (residual)

$A(t) = t - S_{N(t)}$: $\Pr\{A(t) \le a\} = ?$

$B(t) = S_{N(t)+1} - t$: $\Pr\{B(t) \le b\} = ?$

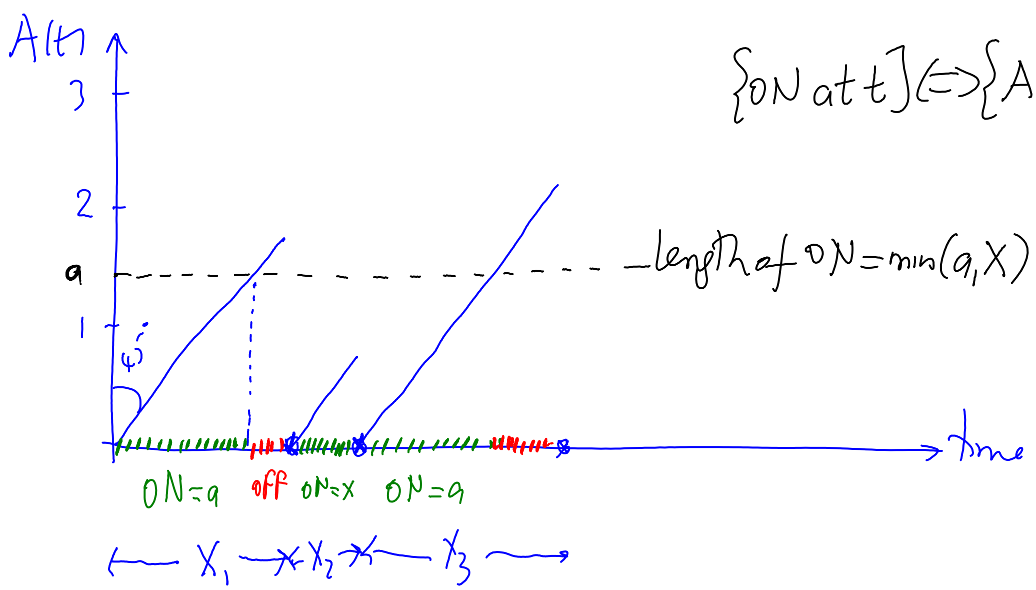
Limiting results as $t \to \infty$



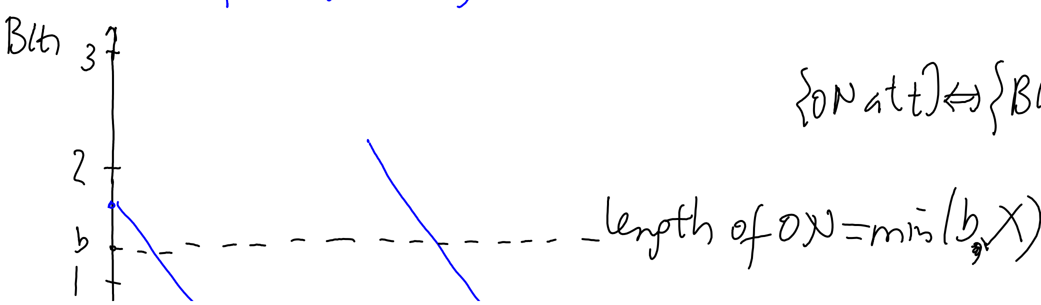
$$\Pr\{A \le a\} = \lim_{t\to\infty} \Pr\{A(t) \le a\}$$

$$\Pr\{B \le b\} = \lim_{t\to\infty} \Pr\{B(t) \le b\}$$

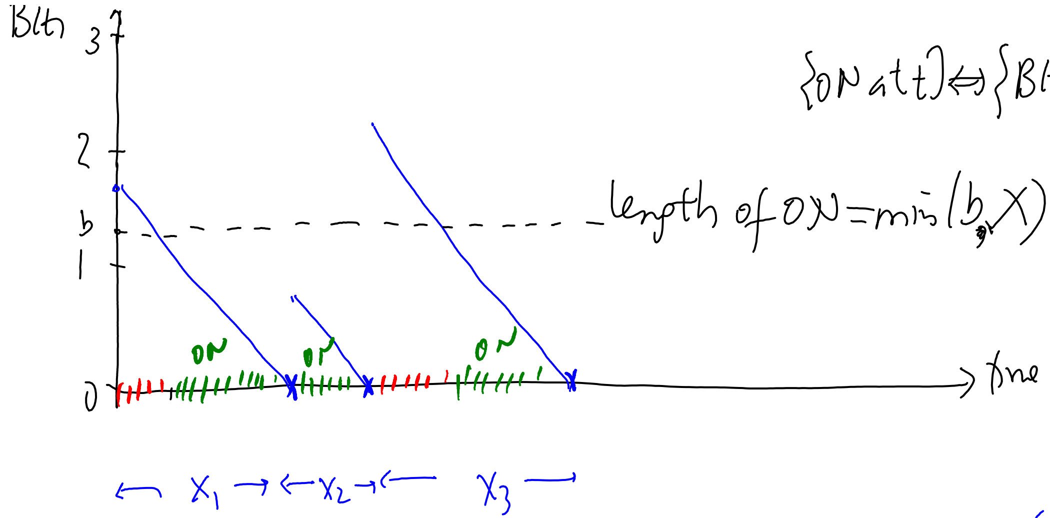
$\{ON \text{ at } t\} \Leftrightarrow \{A(t) \le a\}$



$\{ON \text{ at } t\} \Leftrightarrow \{B(t) \le b\}$



$\{ON \text{ at } t\} \Leftrightarrow \{B(t) \le b\}$



$\mu = E(X)$

$$G_A(a) = Pr\{A \le a\} = \lim_{t\to\infty} Pr\{A(t) \le a\}$$

$$\text{ARP} \rightarrow \; = \frac{E(ON)}{E(X)} = \frac{E[\min(X, a)]}{\mu}$$

$$\min(X, a) = \begin{cases} X, & \text{if } X \le a \\ a, & \text{if } X > a \end{cases}$$

$$E[\min(X, a)] = \int_0^a x\,dF(x) + \int_a^\infty a\,dF(x)$$

show $\rightarrow$ $= \int_0^a \bar{F}(x)\,dx$

$$\Rightarrow \quad G_A(a) = \frac{1}{\mu}\int_0^a \bar{F}(x)\,dx$$

$$g_A(a) = \frac{1}{\mu}\bar{F}(a)$$

show

$$g(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

$$g'(x) = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,dt + b'(x) f(x, b(x)) - a'(x) f(x, a(x))$$

$$\Rightarrow \quad G_A(a) = \frac{1}{\mu}\int_0^a \bar{F}(x)\,dx$$

$$g(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

$$g'(x) = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,dt + b'(x) f(x,b(x)) - a'(x) f(x,a(x))$$

Leibnitz's rule

$$g_A(a) = \frac{1}{\mu}\bar{F}(a)$$

show

$$E(A) = \frac{E(X^2)}{2\mu} = \frac{\sigma^2+\mu^2}{2\mu}$$

Ex. Poisson $\quad g_A(a) = \lambda e^{-\lambda a}, \quad E(A) = \frac{1}{\lambda} = \mu$

$$H_B(b) = Pr\{B \le b\} = \lim_{t\to\infty} Pr\{B(t) \le b\}$$

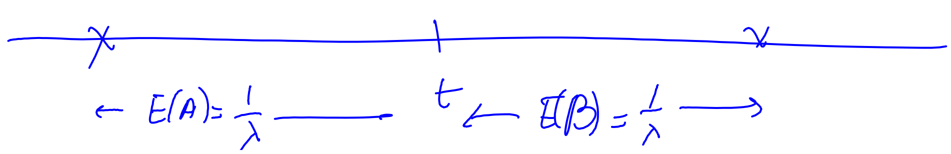
$$= \frac{E(\theta N)}{\mu} = \frac{E[\min(X,b)]}{\mu} = \frac{1}{\mu}\int_0^b \bar{F}(x)\,dx$$

Same as before

$$h_B(b) = \frac{1}{\mu}\bar{F}(b) = \frac{1}{\mu}\int_b^\infty f(x)\,dx$$

$$E(B) = \frac{\sigma^2+\mu^2}{2\mu^2}$$ Same as before

Ex. Poisson $\quad h_B(b) = \lambda e^{-\lambda b}, \quad E(B) = \mu = \frac{1}{\lambda}$



$$E(X) = \frac{1}{\lambda}$$