2013-09-23

Monday, September 23, 2013
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2. Continuous r.v.'s

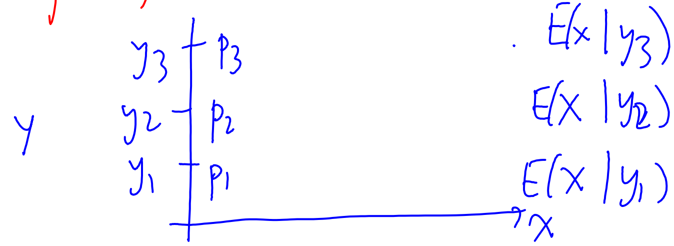
$$f(x,y) = f_{X,Y}(x,y)$$

Here is a better, and more practical, explanation of the conditional densities using an example of [throwing darts](http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/ch-01.html).

Pasted from <http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/ch-01.html>

Very Important property

$E(X)$: tough



$$E(X) = p_1 E(X \mid y_1) + p_2 E(X \mid y_2) + p_3 E(X \mid y_3)$$

$$E(X) = E\big(E(X \mid Y)\big)$$

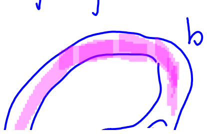
Thm.

$$E(X) = \sum_y E(X \mid Y=y) \Pr(Y=y)$$

$$= \int_0^\infty E(X \mid Y=y) f_Y(y)\, dy$$

Proof. Do it! (Double int's)

Ex. Thief of Baghdad



$\Pr(\text{select any door}) = \frac{1}{3}$

$E(X)$, $Var(X)$ | m.g.f

X: time until freedom

Y: door initially chosen

$Pr(Y=y) = \frac{1}{3}$, $y = 1,2,3$

mgf: $\phi(t) = E(e^{tX}) = \sum_y \underbrace{E(e^{tX} \mid Y=y)} \cdot Pr(Y=y)$ ✓

$Y=1$: $\quad X = a + X'$; X': additional time until freedom has $\underline{\text{same}}$ distrib. as X

$\therefore E(e^{tX} \mid Y=1) = E(e^{t(a+X')}) = e^{ta} E(e^{tX'}) = e^{ta}\phi(t)$

$Y=2$: $\quad X = b + X'$

$\therefore E(e^{tX} \mid Y=2) = e^{tb}\phi(t)$

$e^{t/c}$

$Y=3$: $\quad X = 0$

$\therefore E(e^{tX} \mid Y=3) = 1$

$\Rightarrow \phi(t) = \frac{1}{3}\left[e^{ta}\phi(t) + e^{tb}\phi(t) + 1\right] \Rightarrow \phi(t) = \dfrac{1}{3 - e^{ta} - e^{tb}}$

$\Rightarrow \; E(X) = \phi'(0) = a + b$

$Var(X) = 2a^2 + 2ab + 2b^2$

· Probabilities by Conditioning

$Pr(E) = ?$ $\qquad E, Y$

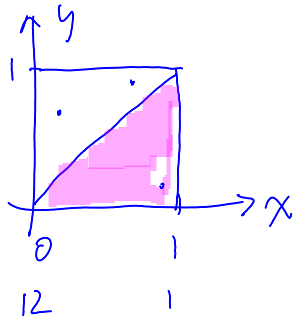
<u>Thm</u>: $Pr(E) = \sum_y Pr(E \mid Y=y) Pr(Y=y)$

$= \int_0^\infty Pr(E \mid Y=y) f_Y(y)\, dy$ $\Big\}$ Show!

·Ex. $X \sim \text{uniform}[0,1]$ $\qquad f_X(x) = 1, \; 0 \le x \le 1$

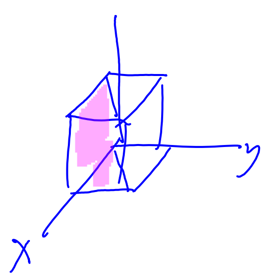
$Y \sim \;\; " \;\; [0,1]$ $\qquad g_Y(y) = 1, \; 0 \le y \le 1$

$E = \{X > Y\}$



X: arrival time of man

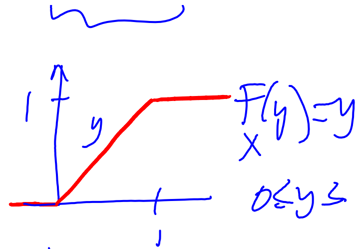
Y: " " " woman



$Pr\{\overbrace{X > Y}^{E}\} = \int_0^1 Pr(X > Y \mid Y = y) g(y)\, dy$

$= \int_0^1 Pr(X > y) \cdot 1\, dy = \int_0^1 [1 - \underbrace{Pr(X \le y)}]\, dy$

$$= \int_0^1 [1-y]\,dy = \frac{1}{2}$$



$$F(y) = \int_0^y g(u)\,du = \int_0^y 1 \cdot du = y$$

$$Pr(X \le a) = F_X(a)$$

$\$_{s_t}$ $\cancel{X_{t}} \; Z_t$

$\$_{s,t}$

2. Exponential distribution & Poisson process

a) Exponential $\qquad X \sim exp(\lambda)$

pdf. $\quad f(x) = \lambda e^{-\lambda x}, \; x \geq 0$

cdf $\quad Pr(X \le x) = \int_0^x f(t)\,dt = 1 - e^{-\lambda x}$ $\quad \Big| \quad \bar{F}(t) = Pr(X > x) = e^{-\lambda x}$

mean $\quad E(X) = \frac{1}{\lambda}$

LT $\quad \tilde{f}(s) = \frac{\lambda}{\lambda + s}$

$$Var(X) = \frac{1}{\lambda^2}, \quad SD(X) = \frac{1}{\lambda}$$

http://www.business.mcmaster.ca/courses/Q711/ChapterComments/documents/Exponential.xls

i) Important properties

T; $\quad f(t) = \lambda e^{-\lambda t}, \quad E(T) = \frac{1}{\lambda} = 1000$ hr

$$\Rightarrow \lambda = \frac{1}{1000}$$

Find

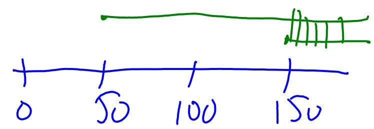
(new) $Pr(T>100) = ?$

(old) $Pr(T>150 \mid T>50) = ?$

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

$$Pr(T>100) = \bar{F}(100) = e^{-\frac{1}{1000}\cdot 100} = .90484$$

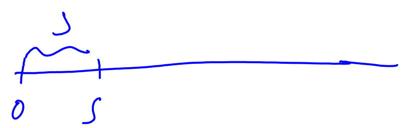
$$Pr(T>150 \mid T>50) = \frac{Pr(T>150, T>50)}{Pr(T>50)}$$



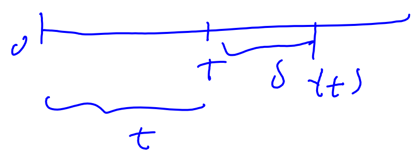
$$= \frac{Pr(T>150)}{Pr(T>50)} = \frac{.86071}{.95123} = .90484$$

This is "memoryless" property

$$Pr(X>t+s \mid X>t) = Pr(X>s)$$

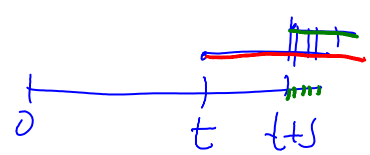


old $\Downarrow$ new $\Downarrow$



$\cap$ and

$$\frac{Pr(X>t+s, X>t)}{Pr(X>t)} = Pr(X>s)$$



$$\bar{F}(t) = Pr(X>t)$$

$$\frac{Pr(X>t+s)}{Pr(X>t)} = Pr(X>s)$$

$x + 3 = 5$

$$\frac{\bar{F}(t+s)}{\bar{F}(t)} = \bar{F}(s)$$

$x + y = 1$

$f'(t) + f(t) = 0$

$h(t) = \int_0^\infty h(t-x) f(x)\,dx$

$\Rightarrow \quad \bar{F}(t+s) = \bar{F}(s)\,\bar{F}(t)$: functional eq'n

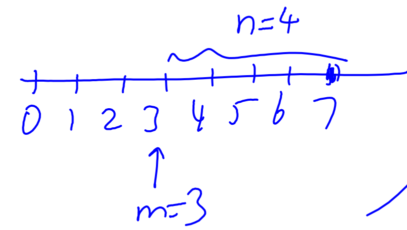
$\Rightarrow$ Sol'n : $f(t) = \lambda e^{-\lambda t}$ old (Ross, p. 24)

If $f(t)$ is not exp., then

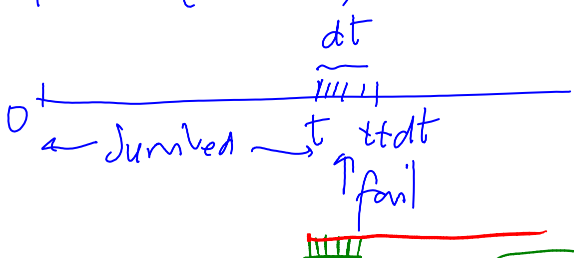
$$Pr\{X > s+t \mid X > t\} = \frac{Pr\{X > s+t\}}{Pr\{X > t\}} = \frac{\bar{F}(s+t)}{\bar{F}(t)}$$

Comment . Discrete

Def $\quad Pr\{N = m+n \mid N > m\} = Pr\{N = n\}$



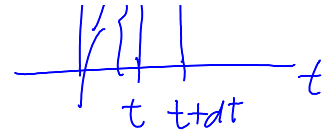
ii) Failure (hazard) rate function



$$Pr\{t \le T \le t+dt \mid T > t\} = \frac{Pr\{t \le T \le t+dt, T > t\}}{Pr\{T > t\}}$$

$$= \frac{Pr\{t \le T \le t+dt\}}{Pr\{T > t\}}$$

$Pr\{T>t\}$



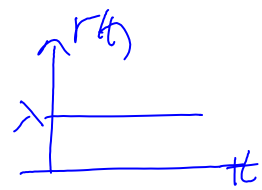
$$\approx \frac{f(t)dt}{1-F(t)} = r(t)dt$$

$f(t)$

$$r(t) = \frac{f(t)}{1-F(t)}$$: hazard rate

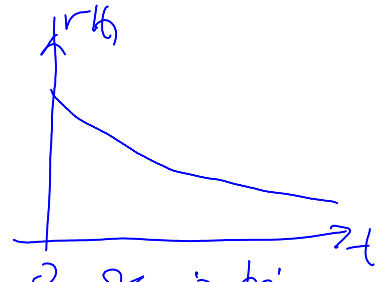
$r(t)dt$

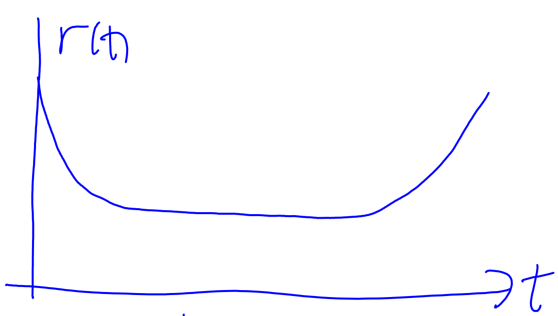
Ex. $f(t) = \lambda e^{-\lambda t}$, $\quad r(t) = \frac{\lambda \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}} = \lambda$



$r(t)dt = \lambda dt$

Ex.



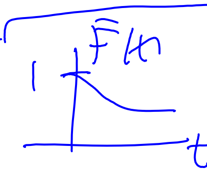


(ii) $r(t)$ uniquely determines $f(t)$ and vice versa

Recall $\quad F(t) = \int_0^t f(u)du \Rightarrow F'(t) = f(t)$

$$\bar F(t) = 1 - F(t) \Rightarrow \bar F'(t) = -f(t)$$

$$\boxed{r(t)} = \frac{f(t)}{\bar F(t)} = \boxed{\frac{-\bar F'(t)}{\bar F(t)}} \Rightarrow \text{DE}: \boxed{\begin{aligned} &\bar F'(t) = -r(t)\bar F(t) \\ &\bar F(0) = 1 \end{aligned}}$$



$$\bar F(t) = e^{-\int_0^t r(u)\,du} \Rightarrow$$

$$f(t) = r(t)\, e^{-\int_0^t r(u)\,du}$$

Ex. $r(t) = \lambda \Rightarrow f(t) = \lambda e^{-\int_0^t \lambda\,du} = \lambda e^{-\lambda t}$

(IV) Erlang (Gamma) r.v.

let $X_1, \dots, X_n$ iid $\exp(\lambda)$

$S = X_1 + \cdots + X_n$: Erlang(n, λ) : Convolution of n $\exp(\lambda)$

$$\tilde f_X(s) = \frac{\lambda}{\lambda + s}, \quad \tilde f_S(s) = [\tilde f_X(s)]^n = \frac{\lambda^n}{(\lambda+s)^n}$$

Item 5 in LT list:

$$\tilde f_S(s) = \lambda^n \frac{1}{(\lambda+s)^{(n-1)+1}} \rightarrow$$

$$\Rightarrow f_S(t) = \lambda^n \frac{1}{(n-1)!} t^{n-1} e^{-\lambda t} = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}, \quad t > 0$$

Show $\quad E(S) = -\tilde{f}_S'(0) = \frac{n}{\lambda}$

$$Var(S) = \frac{n}{\lambda^2}$$

Comment if n real, $(n-1)! \to \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

Comment $\quad X_i \sim exp(\lambda_i)$, $i=1,\dots,n$

$S = X_1 + \cdots + X_n$: generalized Erlang

$$\boxed{1} \to \boxed{2} \to \cdots \to \boxed{n} \to$$

$$1/\lambda_1 \qquad 1/\lambda_2 \qquad\qquad 1/\lambda_n$$

b) Poisson

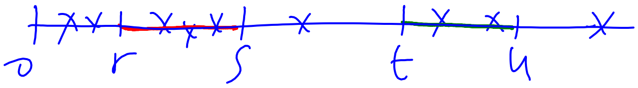
$\{N(t), t \geq 0\}$: Counting process whose
$N(t)$: total # events occurred until t, i.e., in $(0,t]$

i) Def. 1

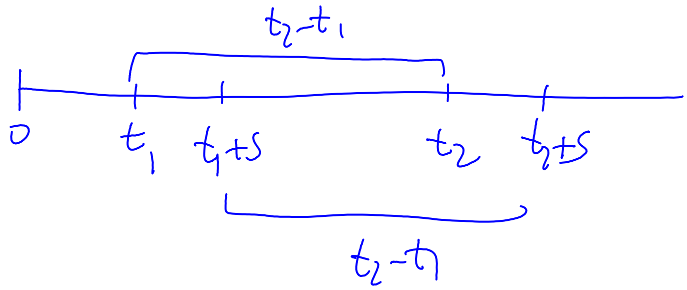
A counting process satisfies,

1) $N(t) \geq 0$
2) $N(t)$ integers
3) $s<t$, $N(s) \leq N(t)$
4) $s<t$, $N(t)-N(s)$: # events in (s,t)

· $N(t)$ has independent increments if # events occurring in disjoint intervals are ind't



- $N(t)$ has stationary increments if distrib. of # events depends only on length of interval, not the position.



ii) Def. 2 . Poisson process is a counting process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$, if,

1) $N(0) = 0$
2) has ind't & stationary increments
3) $Pr\{N(h) = 1\} = \lambda h + o(h)$, small h
4) $Pr\{N(h) \geq 2\} = o(h)$

Thm. Def 2 implies:

$$Pr\{\overbrace{N(t+s) - N(s)}^{N(t)} = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$