```
> restart: # Nash-Stackelberg.mw
```

```
> a:=9/10; b:=9/10;
```

$$a := \frac{9}{10}$$

$$b := \frac{9}{10} \qquad (1)$$

```
> lambda:=1/30; mu:=1/20;
```

$$\lambda := \frac{1}{30}$$

$$\mu := \frac{1}{20} \qquad (2)$$

```
> s:=[15,9]; c:=[8,5];
```

$$s := [15, 9]$$

$$c := [8, 5] \qquad (3)$$

```
> f:=x->lambda*exp(-lambda*x);
```

$$f := x \rightarrow \lambda\, e^{-\lambda x} \qquad (4)$$

```
> g:=y->mu*exp(-mu*y);
```

$$g := y \rightarrow \mu\, e^{-\mu y} \qquad (5)$$

```
> int(f(x),x=0..infinity); int(g(y),y=0..infinity);
```

$$1$$

$$1 \qquad (6)$$

```
> B:='((u-x)/b)+v';
```

$$B := \frac{u - x}{b} + v \qquad (7)$$

```
> A:='((v-y)/a)+u';
```

$$A := \frac{v - y}{a} + u \qquad (8)$$

```
> # J[1]
```
```
> T[1,1]:=unapply('s[1]'*Int('x*f(x)',x=0..u),u,v);
```

$$T_{1,1} := (u, v) \rightarrow s_1 \left(\int_0^u x f(x)\, dx \right) \qquad (9)$$

```
> value(T[1,1](u,v));
```

$$450 - 450\, e^{-\frac{1}{30} u} - 15\, e^{-\frac{1}{30} u} u \qquad (10)$$

```
> T[1,2]:=unapply('s[1]'*u*Int('f(x)',x=u..infinity),u,v);
```

$$T_{1,2} := (u, v) \rightarrow s_1\, u \left(\int_u^\infty f(x)\, dx \right) \qquad (11)$$

```
> value(T[1,2](u,v));
```

$$15\,e^{-\frac{1}{30}u}\,u \tag{12}$$

```
> T[1,3]:=unapply('s[1]'*Int(Int('b*(y-v)*g(y)*f(x)',y=v..'B'),x=
  0..u),u,v);
```

$$T_{1,3} := (u,v) \rightarrow s_1 \left(\int_0^u \int_v^B b\,(y-v)\,g(y)\,f(x)\,\mathrm{d}y\,\mathrm{d}x \right) \tag{13}$$

```
> value(T[1,3](u,v));
```

$$270\,e^{-\frac{1}{20}v} + \frac{2835}{2}\,e^{-\frac{1}{18}u-\frac{1}{20}v} + \frac{45}{2}\,e^{-\frac{1}{18}u-\frac{1}{20}v}\,u - \frac{3375}{2}\,e^{-\frac{1}{20}v-\frac{1}{30}u} \tag{14}$$

```
> T[1,4]:=unapply('s[1]'*Int(Int('(u-x)*g(y)*f(x)',y='B..
  infinity'),x=0..u),u,v);
```

$$T_{1,4} := (u,v) \rightarrow s_1 \left(\int_0^u \int_B^\infty (u-x)\,g(y)\,f(x)\,\mathrm{d}y\,\mathrm{d}x \right) \tag{15}$$

```
> value(T[1,4](u,v));
```

$$-\frac{45}{2}\,e^{-\frac{1}{18}u-\frac{1}{20}v}\,u - \frac{2025}{2}\,e^{-\frac{1}{18}u-\frac{1}{20}v} + \frac{2025}{2}\,e^{-\frac{1}{20}v-\frac{1}{30}u} \tag{16}$$

```
> T[1,5]:=(u,v)->-c[1]*u;
```

$$T_{1,5} := (u,v) \rightarrow -c_1\,u \tag{17}$$

```
> J[1]:=unapply(add(value(T[1,i](u,v)),i=1..5),u,v);
```

$$J_1 := (u,v) \rightarrow 450 - 450\,e^{-\frac{1}{30}u} + 270\,e^{-\frac{1}{20}v} + 405\,e^{-\frac{1}{18}u-\frac{1}{20}v} - 675\,e^{-\frac{1}{20}v-\frac{1}{30}u} - 8\,u \tag{18}$$

```
> J[1](u,v);
```

$$450 - 450\,e^{-\frac{1}{30}u} + 270\,e^{-\frac{1}{20}v} + 405\,e^{-\frac{1}{18}u-\frac{1}{20}v} - 675\,e^{-\frac{1}{20}v-\frac{1}{30}u} - 8\,u \tag{19}$$

```
> J[1,u]:=diff(J[1](u,v),u);
```

$$J_{1,u} := 15\,e^{-\frac{1}{30}u} - \frac{45}{2}\,e^{-\frac{1}{18}u-\frac{1}{20}v} + \frac{45}{2}\,e^{-\frac{1}{20}v-\frac{1}{30}u} - 8 \tag{20}$$

```
> # J[2]
> T[2,1]:=unapply('s[2]'*Int('y*g(y)',y=0..v),u,v);
```

$$T_{2,1} := (u,v) \rightarrow s_2 \left(\int_0^v y\,g(y)\,\mathrm{d}y \right) \tag{21}$$

```
> value(T[2,1](u,v));
```

$$180 - 180\,e^{-\frac{1}{20}v} - 9\,e^{-\frac{1}{20}v}\,v \tag{22}$$

```
> T[2,2]:=unapply('s[2]'*v*Int('g(y)',y=v..infinity),u,v);
```

$$T_{2,2} := (u, v) \rightarrow s_2\, v \left(\int_v^\infty g(y)\, \mathrm{d}y \right) \tag{23}$$

```
> value(T[2,2](u,v));
```

$$9\, \mathrm{e}^{-\frac{1}{20}\,v}\, v \tag{24}$$

```
> T[2,3]:=unapply('s[2]'*Int(Int('a*(x-u)*f(x)*g(y)',x=u..'A'),y=
  0..v),u,v);
```

$$T_{2,3} := (u, v) \rightarrow s_2 \left(\int_0^v \int_u^A a\, (-u + x)\, f(x)\, g(y)\, \mathrm{d}x\, \mathrm{d}y \right) \tag{25}$$

```
> value(T[2,3](u,v));
```

$$243\, \mathrm{e}^{-\frac{1}{30}\,u} + \frac{85293}{49}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u} - \frac{243}{7}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u}\, v - \frac{97200}{49}\, \mathrm{e}^{-\frac{1}{20}\,v - \frac{1}{30}\,u} \tag{26}$$

```
> T[2,4]:=unapply('s[2]'*Int(Int('(v-y)*f(x)*g(y)',x='A..
  infinity'),y=0..v),u,v);
```

$$T_{2,4} := (u, v) \rightarrow s_2 \left(\int_0^v \int_A^\infty (v - y)\, f(x)\, g(y)\, \mathrm{d}x\, \mathrm{d}y \right) \tag{27}$$

```
> value(T[2,4](u,v));
```

$$\frac{243}{7}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u}\, v - \frac{131220}{49}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u} + \frac{131220}{49}\, \mathrm{e}^{-\frac{1}{20}\,v - \frac{1}{30}\,u} \tag{28}$$

```
> T[2,5]:=(u,v)->-c[2]*v;
```

$$T_{2,5} := (u, v) \rightarrow -c_2\, v \tag{29}$$

```
> J[2]:=unapply(add(value(T[2,i](u,v)),i=1..5),u,v);
```

$$J_2 := (u, v) \rightarrow 180 - 180\, \mathrm{e}^{-\frac{1}{20}\,v} + 243\, \mathrm{e}^{-\frac{1}{30}\,u} - \frac{6561}{7}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u} + \frac{4860}{7}\, \mathrm{e}^{-\frac{1}{20}\,v - \frac{1}{30}\,u} \tag{30}$$

$$- 5\, v$$

```
> J[2](u,v);
```

$$180 - 180\, \mathrm{e}^{-\frac{1}{20}\,v} + 243\, \mathrm{e}^{-\frac{1}{30}\,u} - \frac{6561}{7}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u} + \frac{4860}{7}\, \mathrm{e}^{-\frac{1}{20}\,v - \frac{1}{30}\,u} - 5\, v \tag{31}$$

```
> J[2,v]:=diff(J[2](u,v),v);
```

$$J_{2,v} := 9\, \mathrm{e}^{-\frac{1}{20}\,v} + \frac{243}{7}\, \mathrm{e}^{-\frac{1}{27}\,v - \frac{1}{30}\,u} - \frac{243}{7}\, \mathrm{e}^{-\frac{1}{20}\,v - \frac{1}{30}\,u} - 5 \tag{32}$$
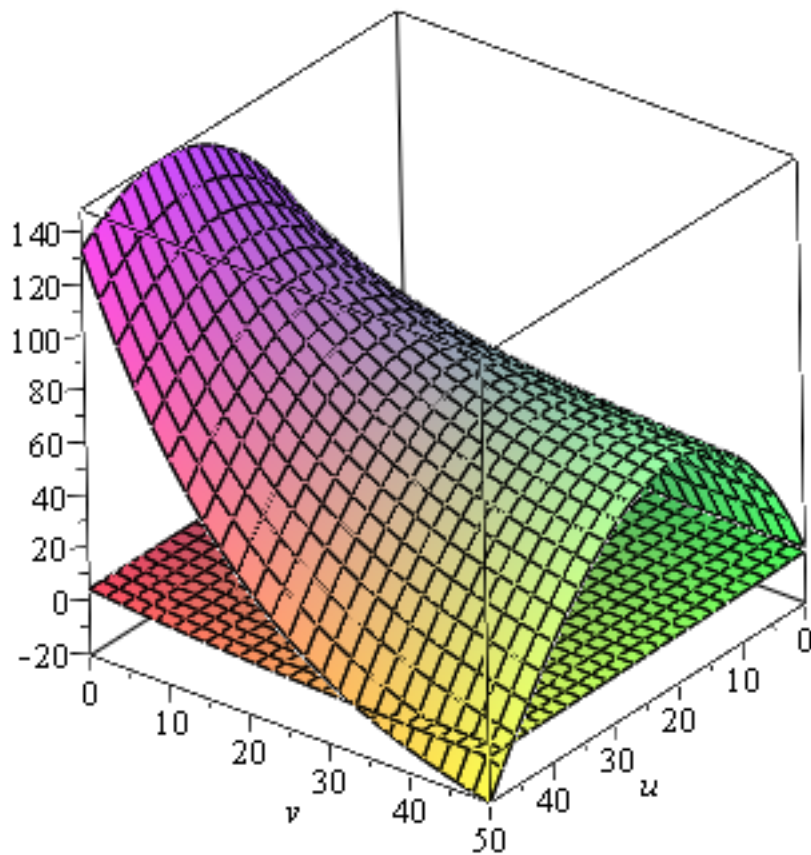
```
> with(plots):
```

```
> implicitplot3d(J[2,v],u=0..50,v=0..50,z=-40..140,shading=none);
```
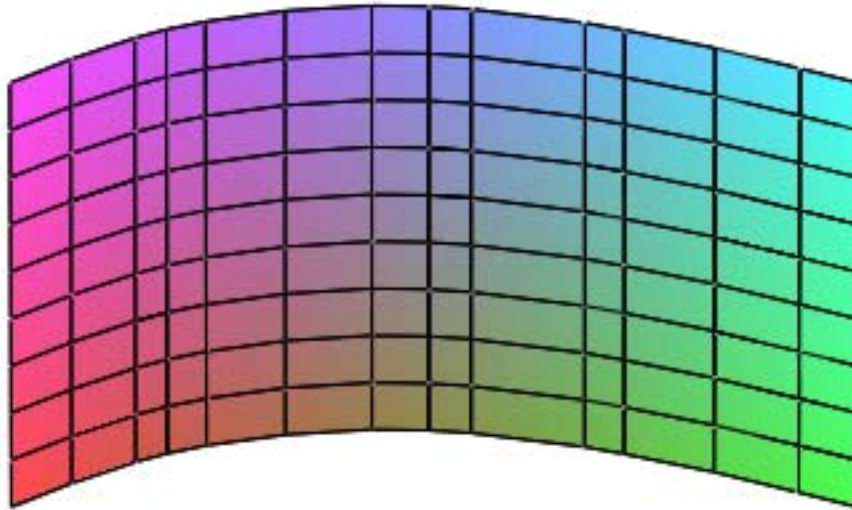
```
> PJ1:=plot3d(J[1](u,v),u=0..50,v=0..50,axes=boxed,shading=none):
> PJ2v:=implicitplot3d(J[2,v],u=0..50,v=0..50,z=-50..140,shading=
  none):
> display(PJ1,PJ2v);
```
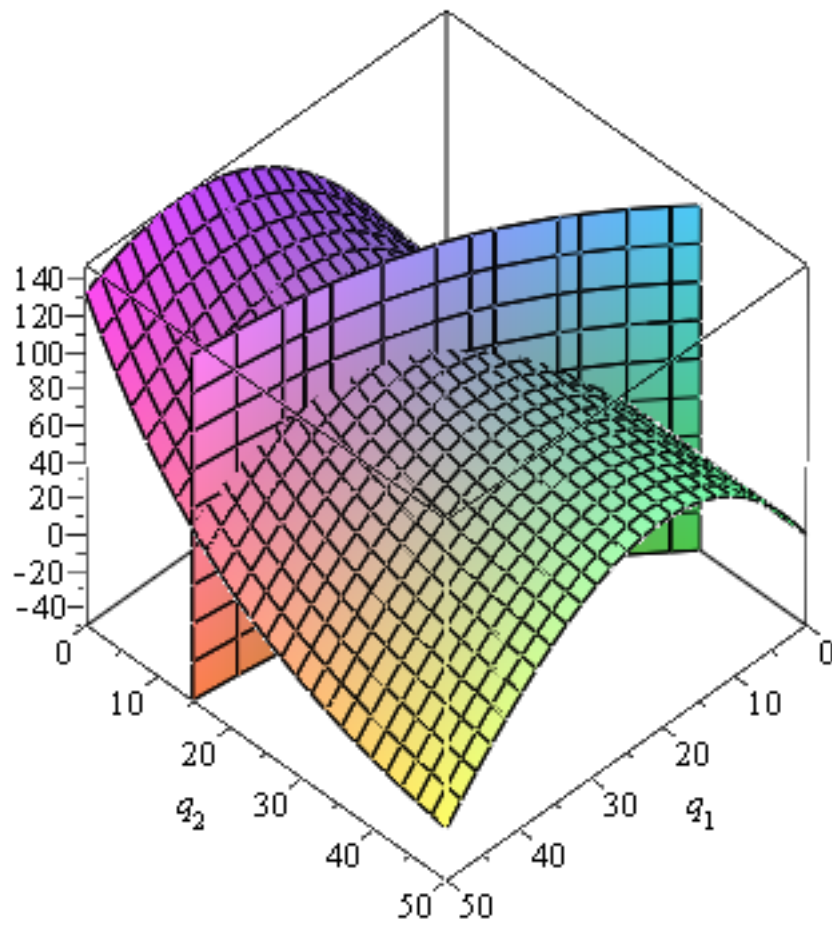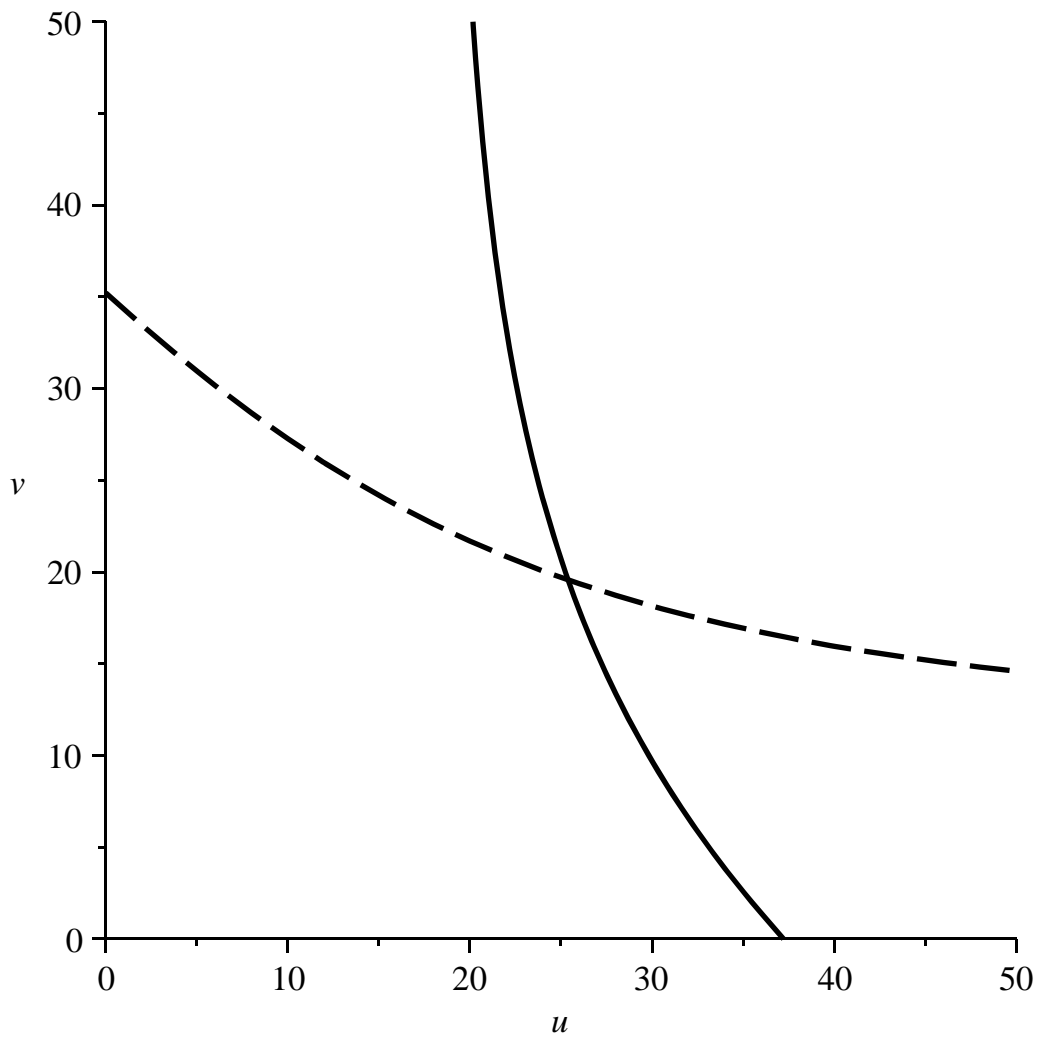
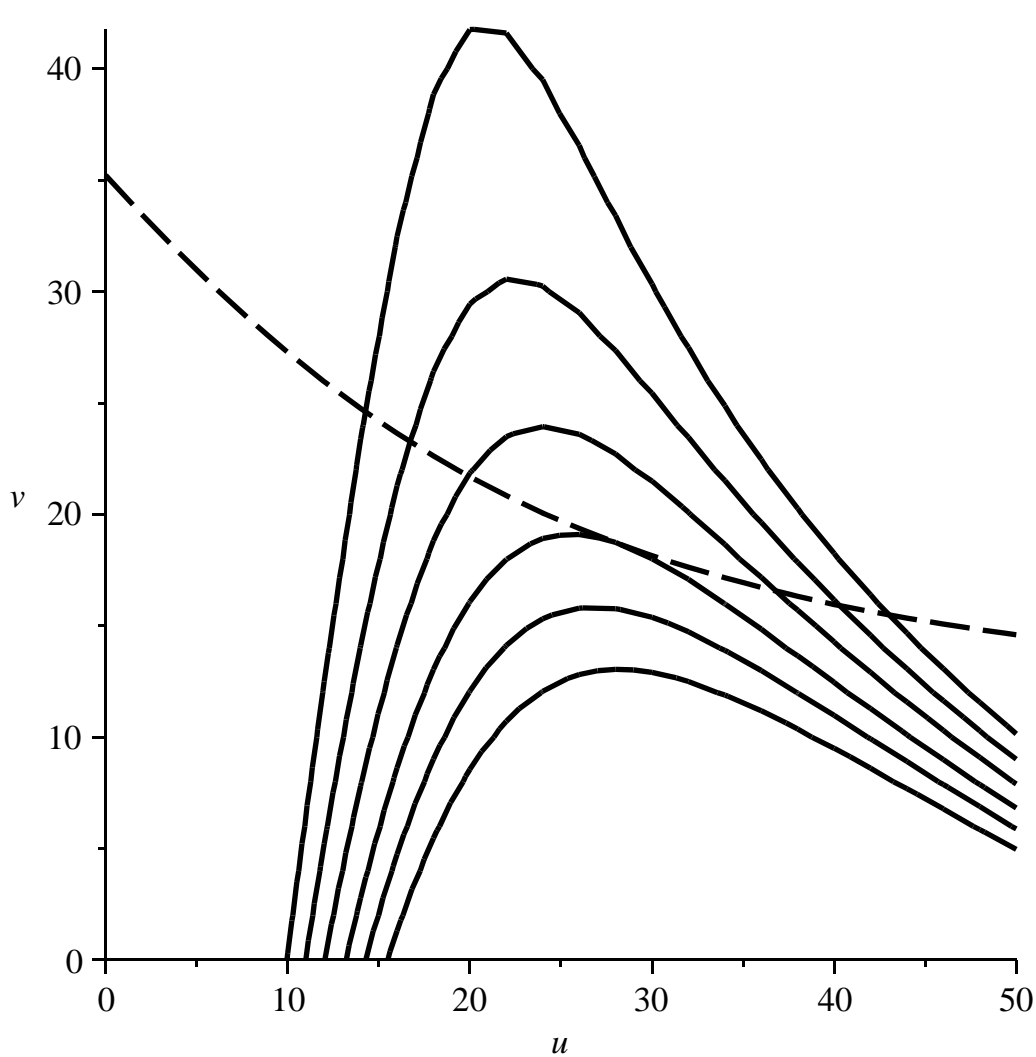> `plot3d([J[1](u,v),J[2,v]],u=0..50,v=0..50,shading=none,axes= boxed);`

```
> CPJ[1]:=contourplot(J[1](u,v),u=0..50,v=0..50,contours=[66,72,
  78,84.3586778,90,96],thickness=2,color=black):
> IPJ[1,u]:=implicitplot(J[1,u],u=0..50,v=0..50,color=red,
  thickness=2,color=black):
> IPJ[2,v]:=implicitplot(J[2,v],u=0..50,v=0..50,color=blue,
  thickness=2,linestyle=dash,color=black):
> display([IPJ[1,u],IPJ[2,v]]);
```

> display([CPJ[1],IPJ[2,v]]);

```
> J[1,u];
```

$$15\, e^{-\frac{1}{30}\,u} - \frac{45}{2}\, e^{-\frac{1}{18}\,u - \frac{1}{20}\,v} + \frac{45}{2}\, e^{-\frac{1}{20}\,v - \frac{1}{30}\,u} - 8 \qquad (33)$$

```
> J[2,v];
```

$$9\, e^{-\frac{1}{20}\,v} + \frac{243}{7}\, e^{-\frac{1}{27}\,v - \frac{1}{30}\,u} - \frac{243}{7}\, e^{-\frac{1}{20}\,v - \frac{1}{30}\,u} - 5 \qquad (34)$$

```
> NashSol:=fsolve({J[1,u],J[2,v]},{u=0..500,v=0..500});
```

$$NashSol := \{u = 25.38764369,\ v = 19.56592045\} \qquad (35)$$

```
> assign(NashSol);
> u,v;
```

$$25.38764369,\ 19.56592045 \qquad (36)$$

```
> J[1](u,v);
```

$$83.6339798 \qquad (37)$$

```
> J[2](u,v);
```

$$35.91337455 \qquad (38)$$

```
> NashResults:=[[u,v],[J[1](u,v),J[2](u,v)]];
```
$$NashResults := [[25.38764369, 19.56592045], [83.6339798, 35.91337455]] \qquad (39)$$

```
> u:='u'; v:='v';
```
$$u := u$$
$$v := v \qquad (40)$$

```
> J[1,u];
```
$$15\,e^{-\frac{1}{30}u} - \frac{45}{2}\,e^{-\frac{1}{18}u - \frac{1}{20}v} + \frac{45}{2}\,e^{-\frac{1}{20}v - \frac{1}{30}u} - 8 \qquad (41)$$

```
> J[2,v];
```
$$9\,e^{-\frac{1}{20}v} + \frac{243}{7}\,e^{-\frac{1}{27}v - \frac{1}{30}u} - \frac{243}{7}\,e^{-\frac{1}{20}v - \frac{1}{30}u} - 5 \qquad (42)$$

```
> J[1](u,v);
```
$$450 - 450\,e^{-\frac{1}{30}u} + 270\,e^{-\frac{1}{20}v} + 405\,e^{-\frac{1}{18}u - \frac{1}{20}v} - 675\,e^{-\frac{1}{20}v - \frac{1}{30}u} - 8\,u \qquad (43)$$

```
> with(Optimization);
```
$$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve] \qquad (44)$$

```
> #NLPSolve(w^3*(v-w)^2+(w-x-1)^2+(x-y-2)^2+(y-z-3)^2, {3*z+2*v-3
  = 0, w+x+y+z <= 5}, assume = nonnegative);
> NLPSolve(J[1](u,v),{J[2,v]=0},assume=nonnegative,maximize);
```
$$[84.3607345588502824, [u = 28.2352640752836344, v = 18.6499223487685484]] \qquad (45)$$

```
> with(GlobalOptimization);
```
$$[GetLastSolution, GlobalSolve, Interactive] \qquad (46)$$

```
> StackSol:=GlobalSolve(J[1](u,v),{J[2,v]=0},u=0..100,v=0..100,
  maximize);
```
$$StackSol := [84.3586778353106581, [u = 28.3875508554282128, v = 18.6044088064478892]] \qquad (47)$$

```
> op(2,StackSol);
```
$$[u = 28.3875508554282128, v = 18.6044088064478892] \qquad (48)$$

```
> assign(%);
> u,v;
```
$$28.3875508554282128, 18.6044088064478892 \qquad (49)$$

```
> J[1](u,v); J[2](u,v);
```
$$84.3586778$$
$$33.94991115 \qquad (50)$$

```
> NashResults;
```
$$[[25.38764369, 19.56592045], [83.6339798, 35.91337455]] \qquad (51)$$

```
> StackelbergResults:=[[u,v],[J[1](u,v),J[2](u,v)]];
```
$$StackelbergResults := [[28.3875508554282128, 18.6044088064478892], [84.3586778, \qquad (52)$$
$$33.94991115]]$$