Asset Characteristics and Multi-Factor Efficiency*

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Abstract

We develop a procedure for deriving systematic factors from characteristics based on maximizing each factor's exposure to a characteristic subject to a given level of factor variance. The resulting characteristic-mimicking portfolios (CMP) price mean asset returns identically as the original characteristics. Whether or not characteristics matter as factor loadings or directly based on non-pecuniary preferences for asset attributes, any multi-factor efficient (MFE) portfolio in the sense of Fama (1996) may be generated from a combination of the tangency portfolio and CMPs for any state variable. Our analysis generalizes Fama's MFE concept to any state variable, not just random state variables affecting future investment opportunities.

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1. Introduction

A large literature has attempted to disentangle whether cross-sectional differences in asset returns are best explained by exposure to systematic factor risk or by asset-specific characteristics. Our paper argues that the distinction between factors and characteristics has no empirical meaning, and that advantages of using one approach over the other are mostly of a technical nature. For a natural choice of a characteristic-mimicking portfolio as a factor, the exposure of each asset to the factor is identical to the asset's characteristic.

Starting with the work of Daniel and Titman (1997), Jagannathan and Wang (1996), and Daniel, Hirshleifer, and Subhramanyam (2001), a substantial literature has differentiated between covariance-risk and characteristics-based explanations for asset returns, with varying results.[1] A recent paper by Kozan, Nagel, and Santosh (2015) has a similar objective to our paper by cautioning against this approach. It argues that it is difficult to separate rational and behavioral explanations of returns by distinguishing factors and characteristics (unless an explicit model is specified that accounts for preferences); in contrast our argument goes further in that distinguishing factors and characteristics, in a meaningful way, is pointless to begin with.

We find that a characteristic-mimicking portfolio (CMP) constitutes a factor that has identical pricing implications as the original characteristic. Vice versa, a factor-mimicking characteristic (FMC) has identical pricing implications as the original factor. This is empirically true no matter how assets are priced, fully rationally, fully behaviorally, or by partially rational and behavioral investors. The choice of CMP is dictated objectively by a procedure that maximizes exposure of the factor portfolio to the underlying characteristic subject to a particular level for the return variance of the mimicking factor.

The result is in part related to the Roll identity (Roll, 1977) which implies that a set of test assets is priced correctly by a set of factors if and only if the maximum Sharpe ratio of the factors equals the maximum Sharpe ratio of the test assets (Grinblatt and Titman, 1987). Thus, it is possible to find factors to explain (in ex post data) any pricing outcome generated by characteristics. Our result, however, is more general in that it also applies when a model does not price the test assets correctly and it further applies when characteristics are not stochastic.

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[1] Recent papers include Hou et al. (2011), Daniel and Titman (2012), Luo and Balvers (2015), and Pukthuanthong and Roll (2014).

In the context of a particular theory, specifying preferences (as Kozan et al., 2015, advocate), our perspective may be viewed as a generalization of Fama (1996). We model rational as well as irrational choice by allowing for "non-pecuniary" preferences that lead to investors caring about particular attributes or characteristics of financial assets which become reflected in asset returns.[2] Investors then logically choose only minimum variance portfolios subject to both a particular mean return as well as subject to a particular level of exposure of the portfolio to the characteristic. In Fama (1996) the exposure constraint pertains only to covariance with a state variable; in our case, the state variable may be deterministic and the characteristic can be – but generally is not – a covariance. Thus, the results of Fama (1996) – that investors hold only multi-factor efficient portfolios – apply with the CMP multi-factor efficient and characteristics priced whether or not investors are "rational".

To evaluate if a characteristic or a related risk factor works better, it is therefore not reasonable to compare their performances in explaining cross-sectional differences in mean returns: either the performances are exactly identical or the characteristic-mimicking factor was obtained ad hoc so that any performance difference is arbitrary. However, characteristics and factors may still be differentiated by their explanatory power for return shocks: while the risk premia for characteristics are constant over time, the risk premia for the CMP loadings are the factor realizations which vary over time. If the CMPs explain a significant part of return variation they are valuable for hedging risk and dominate the comparable characteristics formulation. Note that the same procedure for comparing factor and characteristic specification works if the original formulation is the risk factor, in which case we can simply use the loadings on this risk factor to serve as the characteristics.

In the following, we first discuss further the literature in Section 2 and then present a model with priced characteristics in Section 3. We show in Section 4 that a mimicking factor has equivalent pricing implications and Section 5 extends the analysis of Fama (1996). In Section 6 we show more generally that factors and characteristics are interchangeable. Section 7 discusses how implications differ depending on the specifics of how the mimicking portfolios are constructed

[2] The distinction between rational and irrational is a matter of semantics here. If one views "acting consistent with objectives" as rational, then any non-pecuniary component of the objective function – even if completely unrelated to aspects of the objective return distribution – must be viewed as involving rational decision making. If any particular behavior cannot be explained as consistent with some (likely non-standard) objective it simply cannot be explained in a systematic way.

and devises a test to distinguish between characteristics and factors based on their explanatory power for variations in returns rather than mean returns. Section 8 provides empirical results regarding the usefulness of factors or characteristic representations and Section 9 concludes.

2. Context

Fama and French (1992) established empirically that the size (average log of market value) characteristic and value (average log of book-to-market ratio) characteristic of stock portfolios explained differences in mean returns across portfolios much better than market factor loadings. Fama and French (1993) then constructed risk factors based on the size and value characteristics and concluded that these mimicking portfolios functioning as risk factors explained average asset returns in accordance with equilibrium pricing models. Their construction of the mimicking portfolios lacked a formal motivation but consisted roughly of taking the return of the smallest 50% of firms minus the return of the biggest 50% of firms in each period as the size-mimicking factor return; and taking the return of the 30% highest book-to-market value firms minus the return of the 30% lowest book-to-market firms as the value-mimicking factor return.

As both size and value characteristics and size and value risk factors separately performed well in explaining average return differences, the natural question became which one performed better. Daniel and Titman (1997) and Jagannathan and Wang (1996) introduced simple approaches for comparing the importance of characteristics and the mimicking risk factors derived from the characteristics. Daniel and Titman sorted portfolios separately by factor loadings and by characteristics and compared the return differences of the sorted portfolios. Jagannathan and Wang added both the factor loadings and the characteristics themselves to a regression explaining cross-sectional differences in mean returns. The regression results determined whether factor loadings drive out the characteristics or vice versa.

Extensive and continuing application of both approaches has led to diverging results, with sometimes characteristics beating factor loadings (e.g., Brennan et al. 1998 and Chordia et al. 2015), at other times factor loadings beating characteristics (e.g., Davis et al. 2000 and Gao 2012), or the results varying by characteristic (Hou et al. 2011). Daniel and Titman (2012) argue that a sharper empirical distinction is needed in creating separate portfolios based on characteristics and factor loadings to accurately identify which works better.

Recent literature has attempted to further clarify the distinction between characteristics and factors. Lin and Zhang (2012) appeal to the production-based asset pricing context in which firm characteristics are related to investment returns and thus naturally represent loadings on investment risk which must relate to return risk. Presupposing an arbitrage pricing context Kozak, Nagel, and Santosh (2015), hereafter KNS, and Pukthuanthong and Roll (2014) argue that, insofar as characteristics do not match the loadings on systematic risk factors, priced characteristics represent near-arbitrage opportunities (very large Sharpe ratios). Thus, the identifying criterion of (priced) characteristics vis-à-vis factor loadings is that they represent potential for high Sharpe ratios. KNS further provide a theoretical model that illustrates that it is not possible to distinguish irrational from rational explanations for market outcomes. They argue that investors irrationally focusing on asset characteristics may cause price deviations, presenting opportunities for rational investors. If the deviations correlate with a systematic risk factor the "arbitrage" will not eliminate much; if the deviations are uncorrelated with systematic risk the arbitrage should eliminate most of the price deviations.

Gao (2011) provides an approach for employing characteristics to model asset covariances based on the similarity of assets in terms of their characteristics. The covariances now perform better than factor loadings in explaining return differences across assets and drive out the characteristics. This supports the view that it is risk factors, although not approximated through factor loadings, which are more relevant than behavioral factors proxied by characteristics. Moskowitz (2003), Connor (2007) and Suh et al. (2014) also provide methods for relating factor loadings (or, similarly, covariances) to characteristics that differ from the approach of Fama and French (1993). Taylor and Verrecchia (2015) show that given delegation of investment both risk factors and individual characteristics will be priced. Chordia, Goyal, and Shanken (2015) contribute to the debate on loadings versus characteristics by focusing on individual stocks rather than portfolios and adjusting for the substantial measurement error bias that results from estimating loadings for individual stocks. They find that characteristics perform relatively better than factor loadings in explaining average return differences.

The intent in the recent literature is to sharpen the distinction between characteristics and factor loadings, whereas our objective in part is the opposite: to emphasize that the distinction between characteristics and factor loadings is immaterial for explanations of mean returns. Characteristics are identical to factor loadings but on a factor that is generally only trivially different (in construction, not by impact as Kogan and Tian, 2015, argue). Because the factor mimicking portfolio is chosen haphazardly, the difference with the characteristic mimicking portfolio is not fundamental, and basing tests on the difference between the two seems beside the point.

Consider the iconic example of the size characteristic. The idea of Fama and French (1993), hereafter FF, was to explain the empirical importance of the size characteristic for average returns from a risk-taking perspective: for whatever reason, smaller-size firms are more exposed to risk. Accordingly, they construct a risk factor as the return of a factor-mimicking portfolio formed, roughly, by holding the 50% smallest firms and shorting the 50% largest firms. Empirically, the resulting "size factor" helps to explains differences in mean returns well. However, why construct the risk factor in this manner? A theoretical motivation for the construction would protect against data mining. If the idea is that smaller firms are more sensitive to risk, why not construct a risk factor such that the sensitivity to this factor is indeed directly related to firm size? The latter is in fact the CMP generated by our approach. The CMP also provides the theoretical justification that it is the minimum variance portfolio with the largest exposure to the size characteristic.

It is not our intent to provide a factor that is simply a variant to the size factor generated by FF. Our key point is that comparing factor loadings and characteristics is mostly meaningless. Trivially the tests initially proposed by Daniel and Titman and Jagannathan and Wang cannot be performed when CMPs are the mimicking portfolios. While other mimicking portfolios, typically generated from ad hoc assumptions, may produce differences between factor loadings and characteristics, these differences are nonessential from a theoretical perspective, even though Kogan and Tian (2015) argue they may nevertheless be essential empirically.

We further want to point out that CMPs can readily be applied for practical purposes. By constructing a mimicking portfolio for a specific characteristic it becomes possible to estimate a future value of a firm's characteristic (before it is observed) from observation of the firm's factor loading on the CMP, which is typically observed at a higher frequency than the characteristic itself.

Mimicking portfolios were first advocated to represent macro risk factors by Breeden (1979), Grinblatt and Titman (1987), and Huberman et al. (1987) and applied to represent consumption risk by Breeden et al. (1989). These mimicking portfolios convert systematic risk tied to realizations of macro-economic variables into tradable asset portfolios with returns that explain average asset returns just as well as the original macro factors. Lamont (2001) devised an alternative construction of mimicking portfolios, "tracking portfolios", to represent expectations of macro variables. As a key application, Kapadia (2011) used this approach to capture distress risk. Ferson et al. (2006) consider the optimal use of mimicking portfolios representing macro risk factors in the context of predictable time variation. Much earlier, Fama (1976, pp. 326-329) pointed out (see also Ferson et al. 1999) that estimates of risk premia based on Fama-MacBeth (1973) regressions, potentially using characteristics, may be interpreted as portfolios of the assets, with the portfolio weights depending on the characteristics. The approach of Fama and French (1993) provides an alternative without formal validation that generates mimicking portfolios representing aggregate risks. Back et al. (2013) empirically consider the performance of both approaches, terming aptly the Fama (1976) mimicking portfolios as "characteristic pure plays."

Not only does our approach mimic an aggregate risk associated with a desired characteristic, it also provides a vehicle for estimating firm-level characteristics based on measuring factor loadings on the mimicking portfolio. As such it is akin to Lamont (2001) in the limited sense that it can be used to provide estimates of unobservable variables. But rather than generating estimates of expectations of macro variables, we use the CMP to provide estimates of *firm-level* characteristics that cannot be observed in real time. In the context of time variation of the characteristics we anticipate combining our approach with the approach of Ferson et al. (2006) to incorporate conditioning information. Jiang, Kan, and Zhan (2015) provide guidance concerning how to deal with the measurement error issues inherent in the use of mimicking portfolios.

3. A Model with Priced Asset Characteristics

To construct a simple model in which asset characteristics may be priced we start from the traditional single-period Sharpe-Lintner CAPM setting, in which the terminal wealth of each investor k is fully consumed ($c_k = w_k$ with w_k the end of period wealth of the investor), the n risky returns have a multivariate normal distribution and a riskfree asset exists, markets are perfect, and investment opportunities and expectations are homogeneous. We introduce one critical additional assumption: investors may care about features of each risky asset unrelated to the ultimate financial payoff generated.

The utility function thus incorporates what we refer to as "non-pecuniary" preferences. We offer two distinct interpretations for such preferences. First, they may be viewed as purely rational preferences for an economically relevant attribute of an asset such as liquidity, or more broadly as a liking or dislike for the underlying activities financed by the asset. Second, the preferences for particular attributes (such as momentum or glamour, or industry) of assets may reflect an irrational perspective, that deviates from objective expectations, regarding the payoff distribution of the asset. The maximization of the investor objective then simply implies that the investor consistently acts according to the irrational perspective. Kogan, Ross, Wang, and Westerfield (2006) show formally how irrational investor choice may be captured by an attribute added to the utility function and Fama and French (2007) also consider the important of investor tastes.

The investment problem of an investor under the aforementioned assumptions is as follows:

$$\underset{\mathbf{s}_k}{Max} \quad E[u^k(w_k, x_k)], \tag{1}$$

Subject to,

$$w_k = \overline{w}_k (1 + r_f + \mathbf{s}_k' \, \mathbf{r}) \ . \tag{2}$$

$$x_k = \mathbf{s}_k' \, \mathbf{x} \ . \tag{3}$$

In equation (1) the utility function is investor specific as reflected by the superscript *k*. Utility depends on w_k, the end of period wealth of the investor, as well as on a non-wealth attribute of the investor's portfolio summarized by x_k, which we will take to be one-dimensional for simplicity. The wealth constraint in equation (2) states that final wealth equals initial wealth $\overline{w}_k$ times the portfolio return which equals the gross riskfree rate $1 + r_f$ plus the excess return equal to the vector of portfolio shares in the risky assets $\mathbf{s}_k$ times the vector of excess returns $\mathbf{r} = \mathbf{R} - r_f \mathbf{1}$ (where $\mathbf{R}$ is the vector of asset returns and $\mathbf{1}$ is an *n*-vector of ones; the prime indicates the vector transpose). Equation (3) provides the portfolio attribute/characteristic x_k as the value-weighted average of the characteristics of the individual assets in the portfolio, the vector of portfolio shares in the risky assets $\mathbf{s}_k$ times the vector of (deterministic) individual asset characteristics $\mathbf{x}$. For simplicity we have set the characteristic of the riskfree asset equal to zero. Note that the sum of the portfolio shares in the risky assets need not be equal to one because the investor may hold the riskless asset.

Substituting equations (2) and (3) into (1), the first-order conditions for the investment choices of investor k are

$$E[u_w^k(w_k, x_k)\,\overline{w}_k\,\mathbf{r}] + E[u_x^k(w_k, x_k)\,\mathbf{x}] = 0, \qquad (4)$$

where subscripts indicate partial derivatives.

Rewrite the first term in equation (4) by applying the definition of covariance and, given the assumption that returns are multivariate normally distributed, applying Stein's Lemma to obtain:

$$\boldsymbol{\mu}\,\frac{-E[u_{ww}^k(w_k, x_k)]}{E[u_w^k(w_k, x_k)]} = Cov(\mathbf{r}, w_k) + \mathbf{x}\,\frac{-E[u_x^k(w_k, x_k)\,]}{\overline{w}_k\;E[u_w^k(w_k, x_k)\,]} \quad , \qquad (5)$$

where we define the vector of expected excess asset returns as $\boldsymbol{\mu} = E(\mathbf{r})$. Further define: $\theta_w^k \equiv -E[u_{ww}^k(w_k, x_k)] / E[u_w^k(w_k, x_k)]$ and $\theta_x^k \equiv -E[u_x^k(w_k, x_k)\,] / \overline{w}_k\; E[u_w^k(w_k, x_k)\,]$, then:

$$\boldsymbol{\mu}\,\theta_w^k = Cov(\mathbf{r}, w_k) + \mathbf{x}\,\theta_x^k\,. \qquad (5')$$

Sum equation (5') over all investors k (which we are able to do because all investors face the same investment opportunities) to obtain

$$\boldsymbol{\mu}\,\theta_w = \overline{w}_m\; Cov(\mathbf{r}, r_m) + \mathbf{x}\,\theta_x\,. \qquad (6)$$

Equation (6) holds given $\theta_w \equiv \sum_{k=1}^{K} \theta_w^k$ and $\theta_x \equiv \sum_{k=1}^{K} \theta_x^k$ and given that the gross market returns equals $1 + r_m \equiv w_m / \overline{w}_m$ with $w_m \equiv \sum_{k=1}^{K} w_k$ and $\overline{w}_m \equiv \sum_{k=1}^{K} \overline{w}_k$. Thus,

$$\boldsymbol{\mu} = (\overline{w}_m / \theta_w)\, Cov(\mathbf{r}, r_m) + (\theta_x / \theta_w)\, \mathbf{x} \quad . \tag{7}$$

The price of market covariance risk is $\overline{w}_m / \theta_w$ and the characteristics premium is θ_x / θ_w. Given the standard definition of simple betas: $\boldsymbol{\beta} = Cov(\mathbf{r}, r_m) / \sigma_m^2$ we can alternatively state

$$\boldsymbol{\mu} = (\overline{w}_m\, \sigma_m^2 / \theta_w)\, \boldsymbol{\beta} + (\theta_x / \theta_w)\, \mathbf{x} \quad , \tag{7'}$$

where now $\overline{w}_m\, \sigma_m^2 / \theta_w$ is the market risk premium. Clearly, in this model the Sharpe-Lintner CAPM does not generally hold. The CAPM alphas are given by $\boldsymbol{\alpha} = (\theta_x / \theta_w)\, \mathbf{x}$ and it then follows from Roll's analysis that the market portfolio is not efficient (unless $\theta_x = 0$ or $\mathbf{x} = 0$). Since $\theta_w > 0$ if utility functions are concave in wealth and since $\mathbf{x}$ is unrestricted, we examine $\theta_x \equiv \sum_{k=1}^{K} \theta_x^k$ with $\theta_x^k \equiv -E[u_x^k(w_k, x_k)\,] / \overline{w}_k E[u_w^k(w_k, x_k)\,]$. The CAPM would continue to hold only if $\theta_x = 0$. This is certainly possible if for some investors $E[u_x^k(w_k, x_k)\,] > 0$, they like the characteristic, whereas for others $E[u_x^k(w_k, x_k)\,] < 0$, they dislike the characteristic. However, generally, if the characteristic matters in the same direction to a sufficient number of investors, it will be priced and the CAPM will fail to hold. Note that "arbitrage" by a subset of investors who are indifferent to the characteristic, $E[u_x^k(w_k, x_k)\,] = 0$, or have preferences against the grain of the representative investor, will not generally be sufficient to force the characteristics premium to zero, $\theta_x = 0$. The reason is that there is no riskless way of taking a particular position in the characteristic. Nevertheless, such investors will generally hold different portfolios from the representative investor and will benefit from the characteristics premium.

In summary, adding non-pecuniary preferences to the CAPM provides a formal specification for asset characteristics to be priced separately from standard risk characteristics. Whether the non-pecuniary preferences are viewed as rational or as presenting an irrational preference for particular assets, the result is the same that a positive (negative) aggregate preference for a characteristic, $\theta_x < 0$ ($\theta_x > 0$), implies a negative (positive) characteristics premium, and mean returns will decrease (increase) in the size of the characteristic. In the following we will start from equation (7) and consider its implications, irrespective of the model employed to generate it.

4. Characteristic-Mimicking Factors

It is always possible to create a "characteristic-mimicking portfolio" (CMP) to function as an additional factor that prices all assets in the same way as the original characteristics, and converts the premium associated with a deterministic set of asset characteristics to a premium for systematic risk associated with a stochastic risk factor.

Define a characteristic-mimicking factor as a portfolio of the risky assets that: (1) maximizes the exposure to the characteristic, subject to (2) a particular portfolio variance. The covariance matrix of the returns of the *N* risky assets is given by a positive definite **Σ.** Notation otherwise is identical to that of the model in section 2.

$$\underset{\mathbf{s}_x}{Max} \; (\mathbf{s}_x' \mathbf{x}), \qquad \text{s. t.} \quad ½\,(\mathbf{s}_x' \boldsymbol{\Sigma}\, \mathbf{s}_x) = \bar{\sigma}^2 \;, \tag{8}$$

where $\mathbf{s}_x$ is the vector of portfolio shares of the characteristic-mimicking portfolio. Given the Lagrangian formulation with multiplier λ, the first-order conditions based on equation (8) become

$$\mathbf{s}_x = (1/\lambda)\, \boldsymbol{\Sigma}^{-1}\, \mathbf{x}\,, \tag{9}$$

which provides the portfolio shares of the zero-investment characteristic-mimicking factor with return $r_x = (1/\lambda)\, \mathbf{r}'\, \boldsymbol{\Sigma}^{-1}\, \mathbf{x}$. Note that the scale as affected by λ is unimportant for the factor choice since we have a zero-investment portfolio.

> **PROPOSITION:** The characteristics formulation, producing equation (7), prices all assets identically as the factor formulation in which the set of characteristics **x** is replaced by a single characteristic-mimicking factor (CMP) with factor return given by $r_x = (1/\lambda)\, \mathbf{r}'\, \boldsymbol{\Sigma}^{-1}\, \mathbf{x}$.

PROOF. Standard derivation of a two-factor model including the market factor generates

$$\boldsymbol{\mu} = g\; Cov(\mathbf{r}, r_m) + h\; Cov(\mathbf{r}, r_x) \;. \tag{10}$$

However, $Cov(\mathbf{r}, r_x) = \mathbf{\Sigma}\, \mathbf{s}_x$. Hence, from equation (9), equation (10) becomes

$$\boldsymbol{\mu} = g\ Cov(\mathbf{r}, r_m) + (h/\lambda)\ \mathbf{x}\ . \tag{11}$$

Comparison to equation (7) shows that $g = \overline{w}_m / \theta_w$ and $h = \lambda\,(\theta_x / \theta_w)$ implies equal pricing. □

It is straightforward to generalize the analysis to include many characteristics and factors, which we omit. The general implication, however, is that for pricing purposes one may replace a particular risk factor by a set of deterministic characteristics, or vice versa a set of deterministic characteristics by an equivalent systematic risk factor. Our perspective here, relating factors to deterministic characteristics, represents an extension of the concept of multi-factor efficiency explored by Fama (1996) as we explore next.

5. Multi-Factor Efficiency with Characteristics

Fama (1996) introduces the concept of multi-factor efficiency in the context of an equilibrium pricing theory such as the Merton model in which investors care about wealth as well as about state variables affecting what can be done with the wealth, investment opportunities. In this context, "efficient" portfolio choice may be defined as maximizing expected return subject to a given level of portfolio variance and a given portfolio covariance with the state variables. In our case, investors care instead about a given exposure to characteristics. The characteristics may include covariance with the state variables as a special case but may include a broader set of attributes. In particular, the state variables captured by the characteristics need not be stochastic in which case covariance with the state variable would not be defined. Thus, we can generalize the concept of the multi-factor minimum variance frontier as follows:

$$\underset{\mathbf{s}}{Max}\ \ \tfrac{1}{2}\,(\mathbf{s}'\,\mathbf{\Sigma}\,\mathbf{s})\,,\ \text{s.t.}\ \ \mathbf{s}'\boldsymbol{\mu} = \overline{\mu}\ \ \text{and}\ \ \mathbf{s}'\,\mathbf{x} = \overline{x}\,, \tag{12}$$

where $\mathbf{x}$ represents the vector of asset covariances with the state variable in Fama (1996), but may be interpreted more broadly in our case. The multi-factor frontier portfolios then become for any investor *k*:

$$\mathbf{s}_k = \phi_k\ \mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \delta_k\ \mathbf{\Sigma}^{-1}\mathbf{x}\ . \tag{13}$$

Here ϕ_k , δ_k are the Lagrange multipliers for the constraints in (12). Thus, these frontier portfolios are a linear combination of investment in standard tangency portfolios, $\mathbf{s}_T = \phi\,\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$, and the characteristic-mimicking portfolio, $\mathbf{s}_x = \delta\,\mathbf{\Sigma}^{-1}\mathbf{x}$. Efficient investors will hold only these portfolios (in different proportions); they do not accept variance other than that related to mean return or exposure to the characteristic. It follows that, in equilibrium, the market portfolio, $\mathbf{s}_m$, is a linear combination of such frontier portfolios and is accordingly also a multi-factor frontier portfolio. Hence, any investor, in different proportions, ends up holding the market portfolio and the characteristic-mimicking portfolio. It also follows that pricing in the original formulation, in which the characteristic is added to the CAPM, is equivalent to pricing in the two factor model with market return and the characteristic-mimicking portfolio return as the factors.

In equation (13), an investor who does not care about the characteristic will have $\delta_k = 0$ and holds only the tangency portfolio. However, since the market portfolio here generally differs from the tangency portfolio, this investors holds in effect a combination of the market portfolio and the characteristic-mimicking portfolio. The investment in the CMP for this investor is actually an "arbitrage" portfolio that takes (limited) advantage of the characteristics premium in returns; limited because the CMP, as opposed to the characteristic itself, carries risk.

It is interesting to note that the covariance of an asset return *i* with the CMP is given by $\sigma_{ix} = \mathbf{s}_i'\,\mathbf{\Sigma}\,\mathbf{s}_x$ and, because $\mathbf{s}_x = \delta\,\mathbf{\Sigma}^{-1}\mathbf{x}$ for the CMP, this implies that $\sigma_{ix} = \delta\,x_i$. Thus, the covariance of the return on any asset *i* with the CMP is proportional to the characteristic of this asset. Likewise it follows that, in comparison to Fama (1996), we replace "covariance with the state variable" by "covariance with the CMP."

6. Conclusion

An overwhelming number of factors and characteristics has been considered for pricing financial assets. Harvey, Liu, and Zhu (2015) distinguish 113 common (systematic) factors and 212 characteristics. We argue here that there is no point in distinguishing factors and characteristics. Essentially, each of the 212 characteristics may be just as well modeled as a systematic risk factor; and each of the 113 systematic factors may be converted to a characteristic. Neither would impact the explanatory power for pricing assets. The difference has no empirical implications of any kind as long as the characteristic-mimicking factor (CMP) is chosen to maximize the factor's exposure to the characteristic for any specific level of factor variance. In this case, the loading/exposure/sensitivity/factor beta of the CMP return for any asset return is exactly equal to the characteristic of the asset.

There are other ways, of course, to produce mimicking factors from a given set of characteristics. Fama and French (1993) prominently generated value and size factors from value and size characteristics of individual firms. Their method was to rank firms from high to low in terms of their characteristics and then utilize the return differences of firms with the high characteristic level compared to firms with the low characteristics level as the mimicking factors for each characteristic (value and size). Apart from such practical considerations as to whether to compare the high and low 30%, or high and low 50% characteristics, there is no theoretical criterion suggesting Fama and French's particular approach and one may think of a host of alternative approaches for creating the mimicking portfolios. This fact, we think, is crucial because there is a reasonable process, with solid underlying foundation, for creating the CMP which leads to the resulting factors as being empirically indistinguishable from the characteristics.

It is our view that, while mimicking factors obtained by alternative methods will be distinguishable from the underlying characteristics, the distinction is an artifact of the assumed mimicking procedure which has little theoretical backing. Any difference found between the pricing impact of the factor as different from the characteristic's pricing impact is therefore an artifact of the arbitrary mimicking process and not a robust feature of asset pricing.

There are some potentially interesting further implications resulting from the CMP approach. For instance, when applied to the impact of liquidity on asset returns, we have theories such as Amihud and Mendelson (1986) which imply an idiosyncratic liquidity premium for assets related to the proportional bid-ask spreads of the assets. This is an instance of assets being priced by a characteristic, with no theoretical role for systematic liquidity risk. However, the CMP approach generates an idiosyncratic-liquidity-mimicking factor that is indistinguishable from a systematic liquidity factor. On the other hand, Pastor and Stambaugh (2003) provide a theory for a specific systematic liquidity factor. Our analysis implies that the two theories may be separated by the specifics of the construction of the systematic liquidity factors. However, it is incorrect to argue that just any findings of a systematic liquidity factor support Pastor and Stambaugh (2003) over Amihud and Mendelson (1986) or vice versa.

We also plan to apply the procedure for obtaining CMP's in reverse: for a given set of factor portfolio weights we can obtain the characteristics that price assets equivalently. Thus, we may find factor-mimicking characteristics (FMC). These may be useful in a variety of applications. For instance, in the liquidity case we may infer liquidity characteristics for any firm based on the Pastor-Stambaugh liquidity factor. This will make it possible to find correlations with other firm-specific liquidity measures. More generally, obtaining characteristics would facilitate applying the approach of Clarke (2015) more broadly. Clarke regresses firm returns on a set of characteristics, and infers expected return based on the results. He then sorts the firms by their expected returns before isolating factors that explain these returns. Being able to generate characteristics from factors will increase the set of variables that can be used for these purposes.

Our ultimate objective is to apply CMPs to learn about underlying firm characteristic from high-frequency market information. For the applications we intend to create CMPs according to eq. (17). Then calculate **B** as in eq. (18) from high-frequency returns and use it to infer the characteristics **X** that are not yet available. To do so accurately we intend to be more specific about stochastic time variation in the characteristics and the loadings and how to efficiently utilize lagged values of the characteristics in conjunction with factor loadings estimated from recent and past observations. We will follow the approach of Ferson et al. (2006) to properly incorporate conditioning information.

We are in the process of applying the approach to continuous *tracking of bank risk* and risk absorption capacity. The characteristics of interest are 46 publicly traded banks' key balance sheet items that measure capital and liquidity as well as risk. The variables involved in measuring characteristics **X** are available from the FDIC's quarterly Call Reports at the bank portfolio level and proprietary information on these banks' securities holdings collected through Federal Reserve Board of Governors FR Y-14 forms (shared with the Office of the Comptroller of the Currency). This information allows us to calculate the CMP weights for any specific characteristic we are interested in as an indicator of risk or liquidity. A simple example is the capital-asset ratio. The loadings, **B**, on the CMP will be estimated from daily CRSP stock returns of these banks traded on NYSE.

We further intend to apply our analysis to *unobservable actions of mutual funds* taken between snapshots of their required holding report dates. Kacperczyk, Sialm, and Zheng (2008) show there is a difference between reported mutual fund returns and returns constructed from mutual fund holdings. The identity between risk loadings and characteristics allows us to attribute this discrepancy to particular trading strategies measured from inferred changes in holdings of particular asset groups (fraction of cash, value stocks, etc.) viewed as characteristics. The holdings information at the portfolio level is available to us from Thomson Reuters' 13f and s12 files in which money managers are required by the SEC to report their stock holdings (when their portfolio's value exceeds $100 million) within 45 days after the last day of each quarter. The returns on these mutual funds are in the CRSP Mutual Funds Quarterly file and constructed from the holdings information in the s12 files. The holding information on stocks, bonds, and cash values will be mapped to TAQ and CRSP daily return files to construct the various characteristics. The liquidity characteristic of mutual fund portfolios can be measured based on Corwin and Schultz (2011), Amihud and Mendelson (1986), and Amihud (2002).

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